Math 117: Homework 5

Due Sunday, November 8, 2026, at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

In Homework 4, Question 5, you considered a sequence s_n satisfying $s_n \neq 0$ for all $n \in \mathbb{N}$ and for which the limit $\lim_{n\to+\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ exists. You showed that, if $L < 1$,then $\lim s_n = 0$.

Now prove that if $L > 1$ (which includes the possibility $L = +\infty$), then $\lim |s_n| = +\infty$.

Hint: Explain why you can apply Homework 4, Question 5, to the sequence $t_n = \frac{1}{|s_n|}$.

Question 2*

Define a sequence s_n as follows: $s_1 = 1$ and, for $n \geq 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$.

(a) Use induction to prove that $s_n \geq 1/2$ for all n.

(b) Use the definition of the sequence and part (a) to show that the sequence is decreasing.

(c) Prove that $\lim s_n = s$ for some $s \in \mathbb{R}$.

(d) Use Homework 4, Question 4(b) to prove that $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} s_{n+1}$.

(e) Use part (d), the definition of s_n, and the limit theorems to find the value of s.

Question 3*

Let s_n be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove σ_n is an increasing sequence.

Question 4*

Suppose the limits of the sequences s_n and t_n exist and $a \in \mathbb{R}$.

(a) Suppose $\lim_{n\to\infty} s_n < a$. Prove that $s_n \geq a$ for at most finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : s_n \geq a\}$ is finite.

(b) Suppose $\lim_{n\to+\infty} t_n > 0$. Prove that there exists $b > 0$ so that $t_n \geq b$ for all but finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : t_n < b\}$ is finite.

(c) Suppose $\lim_{n\to+\infty} s_n t_n > 0$. If $\lim_{n\to+\infty} t_n > 0$, prove that $s_n \leq 0$ for at most finitely many $n \in \mathbb{N}$—in other words, prove that the set $\{n \in \mathbb{N} : s_n \leq 0\}$ has finitely many elements.

Question 5

Suppose $\lim t_n > 0$ and $\lim s_n = +\infty$ Prove that $\lim s_n t_n = +\infty$. (Hint: Use Question 4.)

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Question 6

Suppose $a < b + \epsilon$ for all $\epsilon > 0$. Prove that $a \leq b$. (Compare to Homework 1, Question 3 and Homework 3, Question 9.)

Question 7*

(a) Given a nonempty subset $S \subseteq \mathbb{R}$ and $k > 0$, let $kS = \{ks : s \in S\}$. Prove that $\sup(kS) = k \sup(S)$. You may use the fact that, for any $k > 0$, $k \cdot (+\infty) = +\infty$ and $(-k) \cdot (+\infty) = -\infty$.

(b) For $k > 0$, prove that $\limsup(ks_n) = k \limsup s_n$.

(c) How does the result of part (a) change when $k < 0$? Prove it.

Question 8

In class, we defined that a sequence s_n of real numbers *converges* to some $s \in \mathbb{R}$ provided that, for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.

Prove that the above definition is equivalent to the following definition: *for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ so that $n \geq N$ ensures $|s_n - s| < \epsilon$.*

Question 9

Prove $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

Question 10

Given a sequence s_n, prove that $\limsup |s_n| < +\infty$ if and only if s_n is bounded.

Question 11*

Determine whether the following statements are true or false. If they are true, prove them. If they are false, give a counterexample and justify it.

(a) If a sequence s_n satisfies $\limsup s_n = 2$, then $s_n > 1.99$ for all n large enough.

(b) If a sequence s_n satisfies $\limsup s_n = b$, then $s_n \leq b$ for all n large enough.

Question 12*

Give examples of...

(a) A sequence x_n of irrational numbers having a limit that is a rational number.

(b) A sequence r_n of rational numbers having a limit that is an irrational number.

Justify your answers.

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Question 13*

Let s_n and t_n be the following sequences that repeat in cycles of four:

$$s_n = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$$
$$t_n = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \dots).$$

Compute the following quantities. You do not need to justify your answer.

(a) $\liminf s_n$

(b) $\limsup s_n$

(c) $\liminf t_n$

(d) $\limsup t_n$

(e) $\liminf s_n + \liminf t_n$

(f) $\liminf s_n + \limsup t_n$

(g) $\limsup s_n + \limsup t_n$

(h) $\limsup(s_n t_n)$

(i) $\liminf(s_n + t_n)$

(j) $\limsup(s_n + t_n)$

(k) $\liminf(s_n t_n)$

Pay attention to the above examples where we see that the liminf and limsup of the sum/product is not necessarily equal to the sum/product of the liminf and limsup.

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