Question 2

Homework 5

(2)

(a) Base case: When $n=1$, $s_1 = 1 \geq \frac{1}{2}$.

Inductive step: Suppose $s_n \geq \frac{1}{2}$. We aim to show $s_{n+1} \geq \frac{1}{2}$. By definition $s_{n+1} = \frac{1}{3}(s_n + 1)$.

Since $s_n \geq \frac{1}{2}$, $s_n + 1 \geq \frac{3}{2}$, so

$$s_{n+1} = \frac{1}{3}(s_n + 1) \geq \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}.$$

This completes the proof.

(b) We aim to show $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$.

By part (a), $s_n \geq \frac{1}{2}$, so $\frac{2}{3} s_n \geq \frac{1}{3}$. Thus, by definition of the sequence,

$$s_{n+1} = \frac{1}{3}(s_n + 1) = \frac{s_n}{3} + \frac{1}{3} \overset{(a)}{\leq} \frac{s_n}{3} + \frac{2 s_n}{3} = s_n,$$

which completes the proof.

(c) Since s_n is a decreasing sequence, $s_1 \geq s_n \ \forall n \in \mathbb{N}$. Since $s_n \geq \frac{1}{2} \ \forall n \in \mathbb{N}$, we have $\frac{1}{2} \leq s_n \leq s_1 = 1 \ \forall n \in \mathbb{N}$. Thus s_n is a bounded, decreasing sequence. Since all bounded monotone sequences converge, $\lim_{n\to\infty} s_n = s$ for some $s \in \mathbb{R}$.

(d) This is an immediate consequence HW4 (6)(b) for $m = 1$.

(e) By part (d) and the limit theorems,

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{1}{3}(s_n + 1) = \frac{1}{3}(s+1).$$

Thus, $\frac{2}{3}s = \frac{1}{3}$, so $s = \frac{1}{2}$.

Updated reference in (d): HW4 Q6(b) is now HW4 Q4(b).

Question 3

Homework 5

(3) We must show

$\sigma_{n+1} = \frac{1}{n+1}(S_1 + S_2 + \dots + S_{n+1}) \geq \frac{1}{n}(S_1 + S_2 + \dots + S_n) = \sigma_n$,

which is equivalent to showing

$(S_1 + S_2 + \dots + S_{n+1}) \geq \frac{n+1}{n}(S_1 + S_2 + \dots + S_n) = (1 + \frac{1}{n})(S_1 + \dots + S_n)$.

$= S_1 + \dots + S_n + \frac{1}{n}(S_1 + \dots + S_n)$.

Subtracting $S_1 + \dots + S_n$ from both sides shows this is equivalent to showing $S_{n+1} \geq \frac{1}{n}(S_1 + \dots + S_n)$. Multiplying both sides by n, this is equivalent to $n S_{n+1} \geq S_1 + \dots + S_n$. Since S_n is increasing, $S_{n+1} \geq S_i \ \forall\, i = 1, \dots, n$ which gives the result.

Question 4

Homework 5

(5)@ First, suppose $\lim_{n\to\infty} s_n = S$ for $S < a$. If we define $\varepsilon = a - S$, then $\varepsilon > 0$.
By definition of convergence, there exists $N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - S| < \varepsilon \Leftrightarrow S - \varepsilon < s_n < S + \varepsilon$

Thus $n > N$ ensures $s_n < S + \varepsilon = a$.
Hence $\{n \in \mathbb{N} : s_n \geq a\} \subseteq \{1, 2, \ldots, \lfloor N \rfloor\}$.
Thus, $\{n \in \mathbb{N} : s_n \geq a\}$ is a finite set.

Next, suppose $\lim_{n\to\infty} s_n = -\infty$. Let $M = \min\{-1, a\}$. Then $\exists$ N s.t. $s_n < M \leq a$ for all $n > N$.
Thus, $\{n \in \mathbb{N} : s_n \geq a\}$ is a finite set.

(b) First, suppose $\lim_{n\to\infty} t_n = t > 0$. Then $\lim_{n\to\infty} t_n > \frac{t}{2}$. Applying part (a) with $s_n = -t_n$ and $a = -\frac{t}{2}$, we obtain that
$\{n \in \mathbb{N} : s_n \geq a\} = \{n \in \mathbb{N} : -t_n \geq -\frac{t}{2}\}$
$= \{n \in \mathbb{N} : t_n \leq \frac{t}{2}\} \supseteq \{n \in \mathbb{N} : t_n < \frac{t}{2}\}$
are all finite sets. This shows the result for $b = \frac{t}{2}$.

Now, suppose $\lim_{n\to\infty} t_n = +\infty$. Then $\exists$ N s.t. $\forall n > N$, $t_n > 1$. Thus, $\{n \in \mathbb{N} : t_n \leq 1\} \supseteq \{n \in \mathbb{N} : t_n < 1\}$ is finite. This shows the result for $b = 1$.

(c) By part (b), $\exists$ $b_1 > 0$ so that $t_n \geq b_1$ for all but finitely many n and $\exists$ $b_2 > 0$ so that $s_n t_n \geq b_2$ for all but finitely many n. Thus, there exists N_1, N_2 so that $n > N_1$ ensures $t_n \geq b_1 \geq 0$ and $n > N_2$ ensures $t_n s_n \geq b_2 \geq 0$. Thus $n > \max\{N_1, N_2\}$ ensures $s_n \geq 0$. This shows $\{n \in \mathbb{N} : s_n \leq 0\}$ has at most $\max\{N_1, N_2\}$ elements.

Type-o in solution to (c): The eventual conclusion is $n > \max\{N_1, N_2\}$ ensures $s_n > 0$. This proves that $\{n : s_n \leq 0\}$ is finite.

(6)

(a) Case 1: $\sup(S) = +\infty$. Assume, for the sake of contradiction, that m is an upper bound for kS. Then $s \leq \frac{m}{k}$ $\forall s \in S$, which is a contradiction. Thus, $\sup(kS) = +\infty$.

Case 2: $\sup(S) \in \mathbb{R}$. Since $\sup(S)$ is an upper bound for S, $s \leq \sup(S)$ $\forall s \in S \Rightarrow ks \leq k\sup(S)$ $\forall s \in S \Rightarrow k\sup(S)$ is an upper bound for kS. Furthermore if m is another upper bound of kS, then $\frac{m}{k}$ is an upper bound of S, so $\sup(S) \leq \frac{m}{k}$ $\Rightarrow k\sup(S) \leq m$. This shows $k\sup(S)$ is the least upper bound of kS.

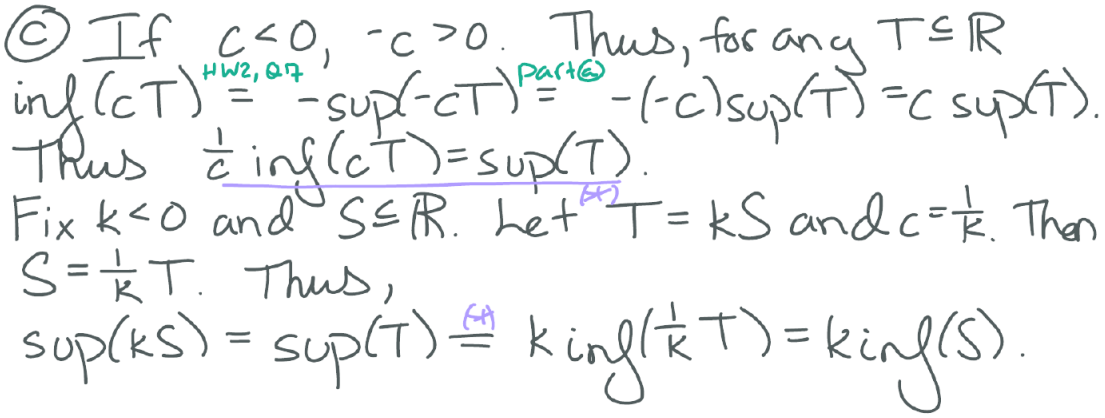
(b) $$\limsup_{n\to\infty}(ks_n) = \lim_{N\to\infty} \sup\{ks_n : n > N\}$$

$$\overset{(a)}{=} \lim_{N\to\infty} k\sup\{s_n : n > N\}$$

$$= k \lim_{N\to\infty} \underbrace{\sup\{s_n : n > N\}}_{a_N}$$

If a_N converges, this is because limit of product is product of limits. If $\lim a_N = +\infty$, it is Q4. If $\lim a_N = -\infty$, it is Q4 and HW4 Q10.

$$= k \limsup_{n\to\infty} s_n$$



Updated references in (b): The old Q4 is now Q5. The scalar-limit theorem in HW4 Q9 justifies pulling k outside the limit, including the cases $\lim a_N = \pm\infty$; use this in place of the old reference to HW4 Q10.

(10) (a) False. Consider: $s_n = (-1)^n 2$.

Then $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$

$= \lim_{N\to\infty} 2 = 2$.

However, all odd elements of s_n are strictly less than 1.99.

(b) False. Consider $s_n = b + \frac{1}{n}$.

Since s_n is convergent,

$\lim_{n\to\infty} s_n = b = \limsup_{n\to\infty} s_n$.

However $s_n > b$ for all n.

(11)(a) Define $x_n = \frac{\sqrt{2}}{n}$. As shown in class, $\sqrt{2}$ is an irrational number. Since $\mathbb{Q}$ is a field, the product of two rational numbers is a rational number. Since $\mathbb{N} \subseteq \mathbb{Q}$ and $x_n \cdot n = \sqrt{2} \notin \mathbb{Q}$, we must have that $x_n \notin \mathbb{Q}$, so x_n is a sequence of irrational numbers.

Claim: $\lim_{n\to\infty} x_n = 0$. We must show that

for all $\varepsilon > 0$, there exists N s.t. $n > N$ ensures $|x_n| < \varepsilon$. Note that
$|x_n| = \left|\frac{\sqrt{2}}{n}\right| = \frac{\sqrt{2}}{n} < \varepsilon \iff \frac{\sqrt{2}}{\varepsilon} < n$.
Therefore, for all $\varepsilon > 0$, if we take $N = \frac{\sqrt{2}}{\varepsilon}$, then for all $n > N$, $|x_n| < \varepsilon$.

(b) Define $r_n = 1.\underbrace{41421\ldots}_{\text{first } n \text{ digits of decimal approximation of } \sqrt{2}}$.

Or more precisely, we define r_n by $r_n = \lfloor \sqrt{2} \cdot 10^n \rfloor / 10^n$, where $\lfloor a \rfloor$ represents the largest integer less than or equal to a. Then $r_n \in \mathbb{Q}$.

Claim: $\lim_{n\to\infty} r_n = \sqrt{2}$. Note that $|r_n - \sqrt{2}| = 10^{-n} |\lfloor \sqrt{2} \cdot 10^n \rfloor - \sqrt{2} \cdot 10^n| \leq 10^{-n}$ and $10^{-n} < \varepsilon \iff \frac{1}{\varepsilon} < 10^n \iff \log_{10} \frac{1}{\varepsilon} < n$. Therefore, for all $\varepsilon > 0$, if we take $N = \log_{10} \frac{1}{\varepsilon}$, then for all $n > N$, $|r_n - \sqrt{2}| < \varepsilon$.

Question 13

Homework 5

(a) $\liminf s_n = 0$.

(b) $\limsup s_n = 2$.

(c) $\liminf t_n = 0$.

(d) $\limsup t_n = 2$.

(e) $\liminf s_n + \liminf t_n = 0 + 0 = 0$.

(f) $\liminf s_n + \limsup t_n = 0 + 2 = 2$.

(g) $\limsup s_n + \limsup t_n = 2 + 2 = 4$.

(h) $\limsup(s_n t_n) = 2$.

(i) $\liminf(s_n + t_n) = 1$.

(j) $\limsup(s_n + t_n) = 3$.

(k) $\liminf(s_n t_n) = 0$.