# Math 117: Homework 6

Due Sunday, November 15, 2026, at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

### Question 1*

Consider the sequences defined as follows:

$$a_n = (-1)^{n+1}, \quad b_n = -\frac{1}{n}, \quad c_n = 2n.$$

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits. Justify your answer.

(c) For each sequence, give its lim inf and lim sup. Justify your answer.

(d) Which of the sequences converges? Diverges to $+\infty$? Diverges to $-\infty$? (You do not need to justify your answer.)

(e) Which of the sequences is bounded? (You do not need to justify your answer.)

### Question 2

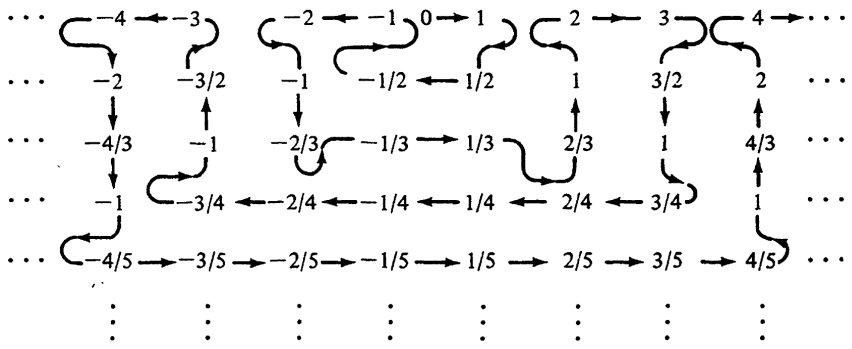
(a) State the definition of convergence for a sequence $s_n$ to a limit $s$.

(b) State what it means for a sequence $s_n$ to *not* converge to a limit $s$ by negating the definition from part (a).

(c) Suppose that $s_n$ does *not* converge to $s \in \mathbb{R}$. Prove that there exists $\epsilon > 0$ and a subsequence $s_{n_k}$ so that $|s_{n_k} - s| \geq \epsilon$ for all $k$.

### Question 3*

One can show that the set of rational numbers $\mathbb{Q}$ can be listed as a sequence $r_n$. The exact procedure is a little tedious, but you can get an idea of how it works by considering the below diagram from the textbook.





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For example, $r_1 = 0, r_2 = 1, r_3 = 1/2$, and so on. Note that some numbers, such as $-1$, are included multiples times.

(a) For any $\epsilon > 0$ and $a \in \mathbb{R}$, show that the set $\{r \in \mathbb{Q} : |r - a| < \epsilon\}$ contains infinitely many elements. (**Hint**: Use denseness of the rationals.)

(b) Let $r_n$ be the sequence of rational numbers. Use part (a) to show that for any $a \in \mathbb{R}$, there exists a subsequence $r_{n_k}$ that converges to $a$. (**Hint**: Use part (a) to show that the set $\{n \in \mathbb{N} : |r_n - a| < \epsilon\}$ is infinite.)

(c) Let $r_n$ be the sequence of rational numbers. Show that there exists a subsequence $r_{n_k}$ satisfying $\lim_{k\to+\infty} r_{n_k} = +\infty$.

**Question 4***

Let $s_n$ be a sequence of nonnegative numbers, and for each $n$ define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

(a) Show $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$.
(**Hint:** For the first inequality, show that $M > N$ implies

$$\inf\{\sigma_n : n > M\} \geq \left(1 - \frac{N}{M}\right) \inf\{s_n : n > N\}.$$

For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \leq \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.)$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example for which $\lim \sigma_n$ exists but $\lim s_n$ does not exist.

**Question 5***

Suppose $s_n$ and $t_n$ are bounded sequences.

(a) Prove that $\limsup s_n + t_n \leq \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences $s_n$ and $t_n$ for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$.


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