

Lecture 1

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Math 117

Prof. Katy Craig

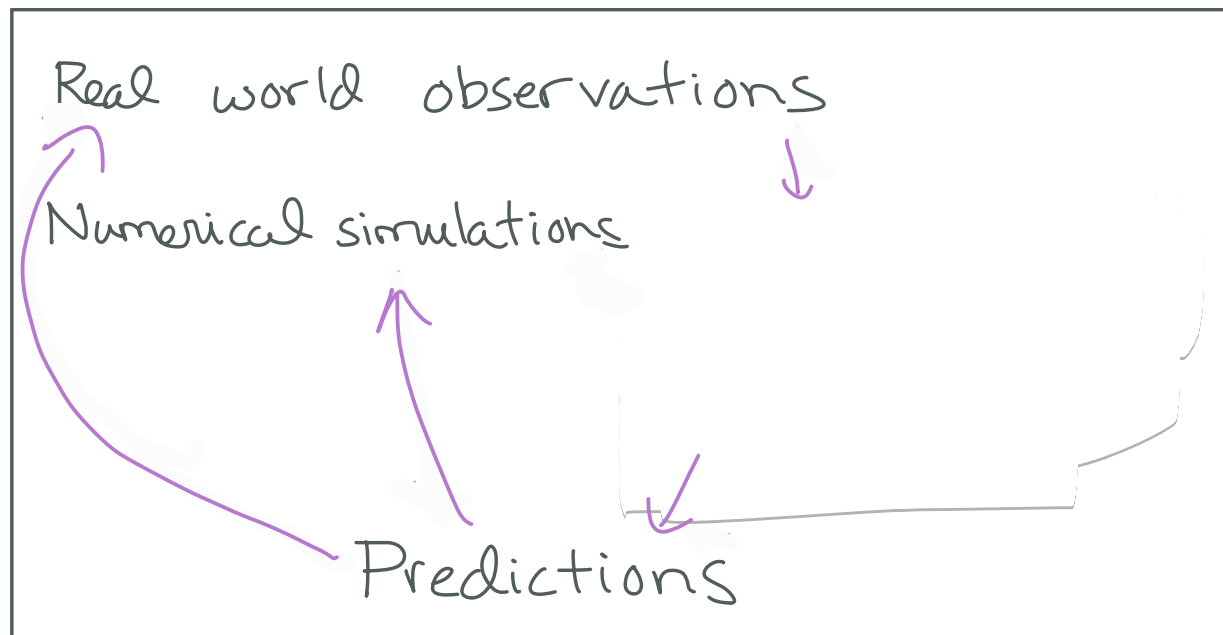
Course goal: transition to higher level math

- "What is a proof?" $\Rightarrow$ "Let's prove interesting things!"
- This is a mathematical writing course.
 - $\hookrightarrow$ You must back up your claims using clear, logical arguments.
 - $\hookrightarrow$ You must be able to precisely state important definitions and theorems.
- If something doesn't make sense...
 - ① Carefully read all relevant definitions and theorems. **Get the textbook!**
 - ② Ask me, TA, or LA for help.
 - ③ Hang in there. If you stay on top of learning definitions and theorems, things will start to make sense. If you don't, things will become more confusing.

Why analysis? What is analysis?

↳ take everything you learned in Calculus and put it on rigorous mathematical footing

↳ analysis is the mathematics of approximation, a key link between mathematics in the real world and mathematics in the fake world



Numbers!

Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

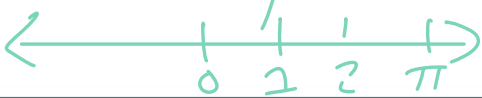
Integers $\mathbb{Z} = \{\dots, -1, 0, 1, 2, \dots\}$

Rational numbers

$$\mathbb{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$$

Real numbers $\mathbb{R} =$ 

Goal of first part of the course
is to define $\mathbb{R}$

Intuitively, $\mathbb{R}$ is all points on
number line: 

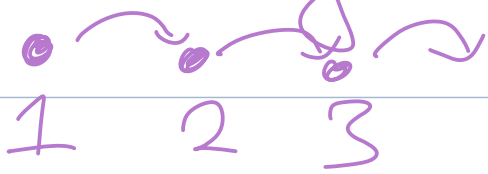
Key properties of these classes of numbers

Property 1: Inductive
characterization of $\mathbb{N}$

If a subset $S \subseteq \mathbb{N}$ satisfies
(i) $1 \in S$
(ii) if $n \in S$ then $n+1 \in S$
then $S = \mathbb{N}$.

Ex: $S = \{1, 3, 5, 7\}$

Mental image: "chain reaction"



Important Remark

This property is the basis of proof by induction.

Suppose $\{P_1, P_2, P_3, \dots\} = \{P_k : k \in \mathbb{N}\}$
is a list of statements.

Ex: $P_k = k$ is a perfect square
 $P_k = \text{Katy wants to eat } k \text{ cookies}$

$P_k = k - 2$ is an integer

If you can prove that

(i) P_1 is true $\rightarrow 1 \in S$

(ii) For any $n \in \mathbb{N}$, if P_n is true, then P_{n+1} is true
 $\rightarrow$ if $n \in S$, then $n+1 \in S$

Then $S = \{k \in \mathbb{N} : P_k \text{ is true}\}$
must satisfy $S = \mathbb{N}$.

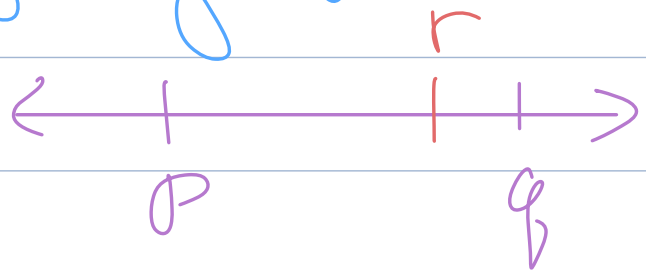
Exercise 1

Prove by induction that, for all $n \in \mathbb{N}$,

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$

Property 2: $\mathbb{Q}$ is dense in $\mathbb{Q}$

Prop: For any $p, q \in \mathbb{Q}$ and $p < q$, there exists $r \in \mathbb{Q}$ satisfying $p < r < q$.



Pl: Let $r = \frac{p+q}{2}$.

First, we show $r \in \mathbb{Q}$. Since $p, q \in \mathbb{Q}$, $\exists m_p, n_p, m_q, n_q \in \mathbb{Z}$ with $n_p, n_q \neq 0$ so that

$$p = \frac{m_p}{n_p}, \quad q = \frac{m_q}{n_q}$$

Thus

$$r = \frac{p+q}{2} = \frac{1}{2} \left(\frac{m_p}{n_p} + \frac{m_q}{n_q} \right)$$
$$= \frac{m_p n_q + m_q n_p}{2 n_p n_q}$$

] integer
] nonzero
integer

so $r \in \mathbb{Q}$.

Finally, since $p < q$, we have $\sqrt{\frac{p}{2}} < \frac{q}{2}$. So

$$p = \frac{p}{2} + \frac{p}{2} < \underbrace{\frac{p}{2} + \frac{q}{2}}_r < \frac{q}{2} + \frac{q}{2} = q.$$

Property 3: Absolute Value and Distance

We may take the absolute value of any number in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Def: $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Thm (basic properties of $|\cdot|$):

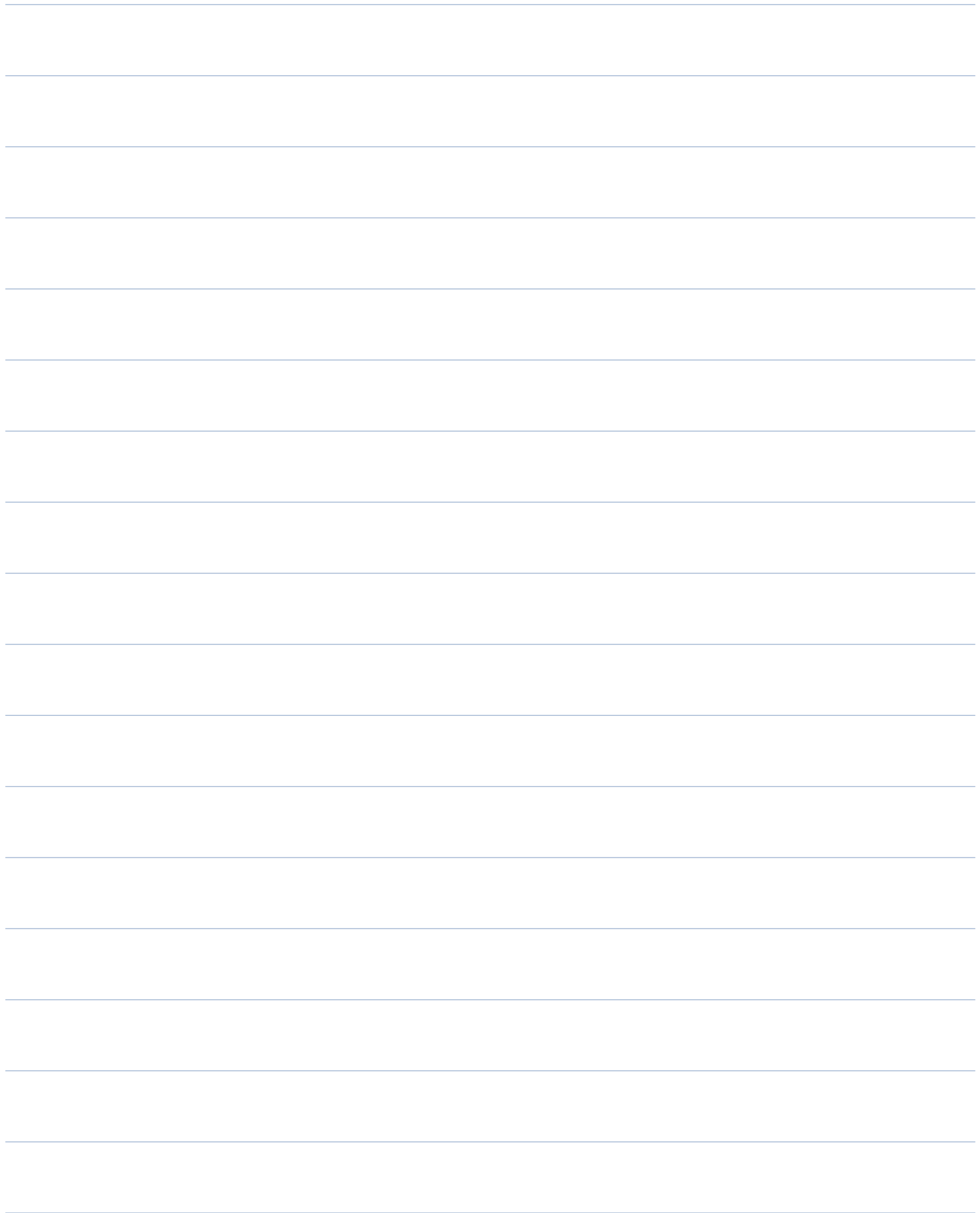
For all $a, b \in \mathbb{R}$,

(i) $|a| \geq 0$

(ii) $|ab| = |a||b|$

(iii) $|a+b| \leq |a| + |b|$

Triangle inequality



Property 4 : Strict Containment

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