

Lecture 6: Highlights

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Thm (limit of sum is sum of limits): If s_n and t_n are convergent sequences, $\lim_{n \rightarrow \infty} (s_n + t_n) = \lim_{n \rightarrow \infty} s_n + \lim_{n \rightarrow \infty} t_n$.

$$s_n = \frac{1}{n}, \quad \cancel{t_n = n^2}$$

Thm (limit of product is product of limits): If s_n and t_n are convergent sequences, $\lim_{n \rightarrow \infty} s_n t_n = \left(\lim_{n \rightarrow \infty} s_n \right) \left(\lim_{n \rightarrow \infty} t_n \right)$

Thm (limit of quotient is quotient of limits): If s_n and t_n are convergent sequences, $s_n \neq 0$ for all n , and $\lim_{n \rightarrow \infty} s_n \neq 0$, then

$$\lim_{n \rightarrow \infty} \left(\frac{t_n}{s_n} \right) = \frac{\lim_{n \rightarrow \infty} t_n}{\lim_{n \rightarrow \infty} s_n}.$$

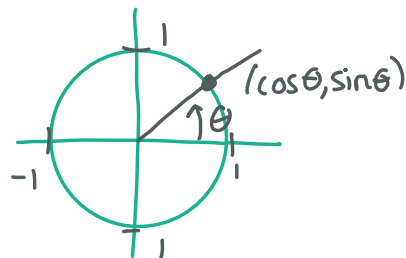
Thm (basic examples):

(a) $\lim_{n \rightarrow \infty} \left(\frac{1}{n} \right)^p = 0$ if $p > 0$

(b) $\lim_{n \rightarrow \infty} a^n = 0$ if $|a| < 1$

(c) $\lim_{n \rightarrow \infty} n^{1/n} = 1$

(d) $\lim_{n \rightarrow \infty} a^{1/n} = 1$ if $a > 0$



Reminder: $\cos(n\pi) = \overset{n=1}{(-1)}, \overset{n=2}{1}, -1, 1, -1, -1, \dots = (-1)^n$
 $\sin(n\pi) = (0, 0, 0, 0, \dots)$

Ex: Find the limit of $S_n = \frac{n-2}{n^2+2}$ and justify your answer.

Pf: Note $S_n = \frac{n-2}{n^2+2} = \frac{\frac{1}{n} - \frac{2}{n^2}}{1 + \frac{2}{n^2}}$. We start by determining limits of numerator and denominator.

Numerator: By Basic Examples thm, $\frac{1}{n} \rightarrow 0$ and $\frac{1}{n^2} \rightarrow 0$. Since the sequence $(-2, -2, -2, \dots)$ converges to -2 , the theorem about limit of product ensures

$$\lim_{n \rightarrow \infty} \frac{-2}{n^2} = \lim_{n \rightarrow \infty} (-2) \frac{1}{n^2} = \lim_{n \rightarrow \infty} (-2) \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0.$$

By thm about limit of sums,

$$\lim_{n \rightarrow \infty} \frac{1}{n} - \frac{2}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n} + \left(\frac{-2}{n^2} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{-2}{n^2} = 0.$$

Denominator: Using thms about limit of sum and product,

$$\lim_{n \rightarrow \infty} 1 + \frac{2}{n^2} = \lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{2}{n^2} = 1 + \lim_{n \rightarrow \infty} (2) \lim_{n \rightarrow \infty} \frac{1}{n^2} = 1 + 2 \cdot 0 = 1.$$

Finally: Using thm about limit of quotient,

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} - \frac{2}{n^2}}{1 + \frac{2}{n^2}} = \frac{\lim_{n \rightarrow \infty} \frac{1}{n} - \frac{2}{n^2}}{\lim_{n \rightarrow \infty} 1 + \frac{2}{n^2}} = \frac{0}{1} = 0. \quad \square$$