

Lecture 7

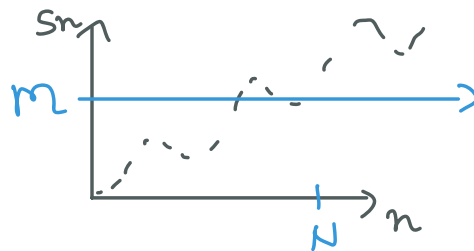
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Reminder: Midterm 1 on
Monday and Wednesday,
Oct 26 and 28

Def (diverges to $+\infty$ or $-\infty$): A sequence s_n diverges to $+\infty$ if for all $M > 0$ there exists $N \in \mathbb{R}$ so that $n > N$ ensures $s_n > M$. We write $\lim_{n \rightarrow \infty} s_n = +\infty$.

Likewise, a sequence s_n diverges to $-\infty$ if for all $M < 0$ there exists N so that $n > N$ ensures $s_n < M$.

We write $\lim_{n \rightarrow \infty} s_n = -\infty$.



Remark:

- If s_n diverges to $\pm\infty$, it does not converge.
- We will say that s_n "has a limit" or "the limit of s_n exists" if either

① s_n converges

② s_n diverges to $\pm\infty$

$$\lim_{n \rightarrow \infty} s_n \in \mathbb{R}$$

$$\lim_{n \rightarrow \infty} s_n \in \{+\infty, -\infty\}$$

Case 1: $\lim_{n \rightarrow \infty} t_n = t, t > 0$
Case 2: $\lim_{n \rightarrow \infty} t_n = +\infty$

Thm: Suppose $\lim_{n \rightarrow \infty} s_n = +\infty$ and $\lim_{n \rightarrow \infty} t_n > 0$.
Then, $\lim_{n \rightarrow \infty} s_n t_n = +\infty$.

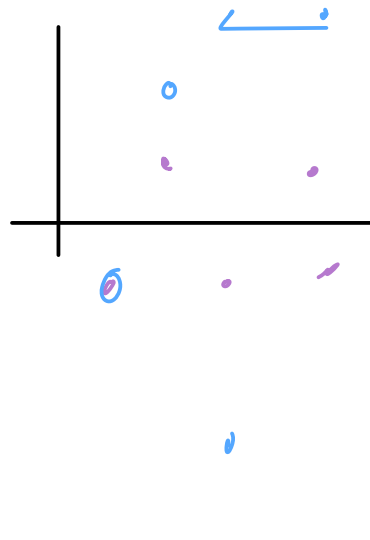
Thm: Suppose s_n is a sequence of positive real numbers. Then $\lim_{n \rightarrow \infty} s_n = +\infty \Leftrightarrow \lim_{n \rightarrow \infty} \frac{1}{s_n} = 0$.

Thm: If $\lim_{n \rightarrow \infty} s_n = +\infty$, then $\lim_{n \rightarrow \infty} (-s_n) = -\infty$.

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Def: (increasing/decreasing/monotone)
 A sequence s_n is **increasing** if $s_n \leq s_{n+1} \forall n$.
 A sequence s_n is **decreasing** if $s_n \geq s_{n+1} \forall n$.
 A sequence s_n is **monotone** if it is either increasing or decreasing.

Thm: All bounded monotone sequences converge



a useful extension...

↙ You can use this stronger version going forward

Thm (limit of quotient): If s_n and t_n are convergent sequences, ~~$s_n \neq 0 \forall n$~~ , and $\lim_{n \rightarrow \infty} s_n \neq 0$, then

will show Wednesday

$$\lim_{n \rightarrow \infty} \frac{t_n}{s_n} = \frac{\lim_{n \rightarrow \infty} t_n}{\lim_{n \rightarrow \infty} s_n}$$

↙ Lemma: If s_n is a convergent sequence and $\lim_{n \rightarrow \infty} s_n \neq 0$, then $\exists N$ s.t. $n > N$ ensures $s_n \neq 0$.

Pf: Assuming Lemma holds, we consider s_n and t_n as sequences

$\lceil N \rceil$ = smallest integer $\geq N$
↓

with domain $\{\lceil N \rceil, \lceil N \rceil + 1, \lceil N \rceil + 2, \dots\}$, for which s_n is always nonzero. By HW4, Q4, s_n and t_n are still convergent and $\lim_{n \rightarrow \infty} s_n \neq 0$.

Applying old version of limit of quotient theorem gives the result.

HW4, Q4

Let s_n be a sequence with domain $\mathbb{N} = \{1, 2, 3, \dots\}$ whose limit exists.

Fix $m \in \mathbb{N}$ and let t_n be a sequence with domain $\{m, m+1, m+2, \dots\}$ defined by $t_n := s_n \quad \forall n \in \{m, m+1, \dots\}$

Then $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n$.