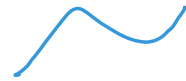


Lecture 8 - Highlights

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Thm: If s_n is an unbounded, increasing sequence, then $\lim s_n = +\infty$. If s_n is an unbounded, decreasing sequence, then $\lim s_n = -\infty$.

In summary, if s_n is monotone

$$\lim_{n \rightarrow \infty} s_n = \begin{cases} s & \text{for } s \in \mathbb{R} \\ +\infty & \text{if } s_n \text{ is bdd} \\ -\infty & \text{if } s_n \text{ is unbdd above} \\ & \text{if } s_n \text{ is unbdd below} \end{cases}$$

Therefore, for any monotone sequence s_n , $\lim s_n$ exists.

Def (limsup/liminf): For any sequence s_n ,

- $\limsup_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} \sup \{s_n : n > N\}$
- $\liminf_{n \rightarrow \infty} s_n = \lim_{N \rightarrow \infty} \inf \{s_n : n > N\}$

Thm: Given a sequence s_n ,

$$\lim_{n \rightarrow \infty} s_n \text{ exists } (\Leftrightarrow) \liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n.$$

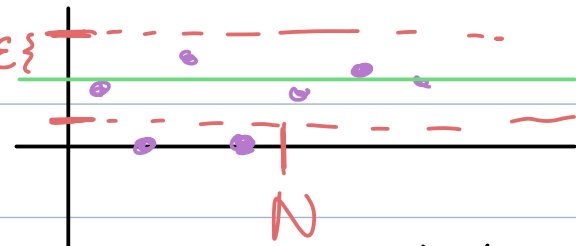
Furthermore, if either of these equivalent conditions holds, $\lim_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n$.

Recall:

we have removed requirement that $t_n \neq 0 \forall n$

Thm (limit of quotient): Given s_n and t_n convergent sequences and $\lim_{n \rightarrow \infty} t_n \neq 0$, then $\lim_{n \rightarrow \infty} \frac{s_n}{t_n} = \frac{\lim_{n \rightarrow \infty} s_n}{\lim_{n \rightarrow \infty} t_n}$.

Lemma: Given a convergent sequence t_n , if $\lim_{n \rightarrow \infty} t_n \neq 0$, then $\exists N$ s.t. $n \geq N$ ensures $t_n \neq 0$.



Pf: Let $t := \lim_{n \rightarrow \infty} t_n$. Since $t \neq 0$, $|t| > 0$, so let $\varepsilon := |t|$. By defn of convergence, $\exists N$ s.t. $n \geq N$ ensures $|t_n - t| < |t|$.

By reverse triangle inequality, $|t| - |t_n| \leq |t_n - t| < |t| \Rightarrow 0 < |t_n|$.

Thus $n \geq N$ ensures $t_n \neq 0$. $\square$

Reverse Δ ineq: $|a - b| \geq |b| - |a|$

Recall from Monday's lecture...

Thm: If $\lim_{n \rightarrow \infty} s_n = +\infty$, then $\lim_{n \rightarrow \infty} (-s_n) = -\infty$.

In fact...

Thm: $\lim_{n \rightarrow \infty} s_n = +\infty$ iff $\lim_{n \rightarrow \infty} (-s_n) = -\infty$

Pl: We showed " $\Rightarrow$ " in Monday's lecture. Assume $\lim_{n \rightarrow \infty} (-s_n) = -\infty$. Fix $m > 0$. Then $-m < 0$. By defn of divergence to $-\infty$, $\exists N$ s.t. $n \geq N$ ensures $-s_n < -m$. Thus $n \geq N$ ensures $s_n > m$. Hence, because $m > 0$ was arbitrary, $\lim_{n \rightarrow \infty} s_n = +\infty$. $\square$