

Question 1

(a) S is bounded above if $\exists M \in \mathbb{R}$ so that $s \leq M$ for all $s \in S$.

(b)

By defn of $A+B$ and the supremum, $\sup(A+B)$ is an upper bound for $A+B$, so $a+b \leq \sup(A+B) \Leftrightarrow a \leq \sup(A+B) - b$ for all $a \in A, b \in B$. Thus, for all $b \in B$, $\sup(A+B) - b$ is an upper bound for A .

(c)

By defn of $\sup(A)$ as the least upper bound for A , $\sup(A) \leq \sup(A+B) - b$ for all $b \in B$. Thus

$$b \leq \sup(A+B) - \sup(A)$$

for all $b \in B$

This shows $\sup(A+B) - \sup(A)$ is an upper bound for B , hence $\sup(B) \leq \sup(A+B) - \sup(A)$.

Question 1 (continued)

This shows $\sup(A) + \sup(B) \leq \sup(A+B)$.

To see the other inequality, note that, since $\sup(A) \geq a \quad \forall a \in A$ and $\sup(B) \geq b \quad \forall b \in B$,

$$\sup(A) + \sup(B) \geq a + b \quad \forall a \in A, b \in B.$$

Thus $\sup(A) + \sup(B)$ is an upper bound for $A+B$, hence

$$\sup(A) + \sup(B) \geq \sup(A+B).$$

Question 2

If $\lim_{n \rightarrow \infty} r_n = -\infty$, the result is immediate.
Thus, it remains to consider the remaining cases.

Case 1: Suppose $\lim_{n \rightarrow \infty} r_n = r \in \mathbb{R}$

Case 1a: If $\lim_{n \rightarrow \infty} t_n = +\infty$, we are done.

Case 1b: Suppose $\lim_{n \rightarrow \infty} t_n = t \in \mathbb{R}$. Assume for the sake of contradiction that $t < r$. Let $\varepsilon = \frac{r-t}{2} > 0$.

Then $\exists N_r, N_t$ s.t. $n > N_r$ ensures $|r_n - r| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon$.

Let $N = \max\{N_r, N_t\}$. Then $n > N$ ensures $t_n < t + \varepsilon = t + \frac{r-t}{2} = \frac{t+r}{2} = r - \frac{r-t}{2} = r - \varepsilon < r_n$.

This contradicts that $r_n \leq t_n \forall n \in \mathbb{N}$.

Thus $\lim_{n \rightarrow \infty} t_n = t \geq r = \lim_{n \rightarrow \infty} r_n$.

Case 1c: Suppose $\lim_{n \rightarrow \infty} t_n = -\infty$. Then $\exists N_t$ s.t. $\forall n > N_t$, $t_n < r - 1$. There also exists N_r s.t. $\forall n > N_r$, $r - 1 < r_n$. Thus for $N = \max\{N_t, N_r\}$, $n > N$ ensures

$$t_n < r - 1 < r_n.$$

Again, this contradicts that $r_n \leq t_n \forall n \in \mathbb{N}$.
Thus $\lim_{n \rightarrow \infty} t_n = -\infty$ is impossible.

Case 2: Suppose $\lim_{n \rightarrow \infty} r_n = +\infty$. Fix $M > 0$. Then $\exists N$ s.t. $\forall n \geq N$, $M < r_n \leq t_n$. This shows $\lim_{n \rightarrow \infty} t_n = +\infty$.

Question 3

Let k, m be positive integers and let $a, b, c, d \in \mathbb{R}$, with $a \neq 0$ and $c \neq 0$.

(a)

Suppose x_n converges and y_n converges, where $\lim_{n \rightarrow \infty} y_n \neq 0$. Then

$$\lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \frac{\lim_{n \rightarrow \infty} x_n}{\lim_{n \rightarrow \infty} y_n}.$$

(b)

Let $M > 0$ and choose

$$N \geq \max \left\{ 1, \frac{M + |d|}{|c|} \right\}.$$

Since m is a positive integer, $n > N$ implies $n^m \geq n$.

If $c > 0$, then for $n > N$, by the reverse triangle inequality,

$$cn^m + d \geq cn - |d| > M.$$

By the definition of divergence to $+\infty$, this proves $cn^m + d \rightarrow +\infty$.

If $c < 0$, then for $n > N$,

$$cn^m + d \leq cn + |d| = -|c|n + |d| < -M.$$

By the definition of divergence to $-\infty$, this proves $cn^m + d \rightarrow -\infty$.

(c)

By part (b), if $c > 0$, then $cn^m + d > 1$ for all sufficiently large n . If $c < 0$, then $cn^m + d < -1$ for all sufficiently large n . In either case, $cn^m + d \neq 0$ for all sufficiently large n .

(d)

Divide the numerator and denominator by n^m :

$$\frac{an^k + b}{cn^m + d} = \frac{an^{k-m} + b/n^m}{c + d/n^m}.$$

Since $m \geq 1$, we have $b/n^m \rightarrow 0$ and $d/n^m \rightarrow 0$, so the denominator on the right converges to $c \neq 0$.

If $k < m$, then $n^{k-m} = 1/n^{m-k} \rightarrow 0$, so the numerator converges to 0. By the quotient theorem,

$$\lim_{n \rightarrow \infty} \frac{an^k + b}{cn^m + d} = \frac{0}{c} = 0.$$

If $k = m$, then $n^{k-m} = 1$, so the numerator converges to a . By the quotient theorem,

$$\lim_{n \rightarrow \infty} \frac{an^k + b}{cn^m + d} = \frac{a}{c}.$$

Question 4

(1) We have $S = (0, 1]$.

- (i) Yes. The maximum is 1, and $\sup(S) = 1$.
- (ii) No. There is no minimum, and $\inf(S) = 0$.

(2)

- (i) **(b)**. The factor n does not have a finite limit, so the product theorem does not apply.
- (ii) **(b)**. Every monotone real sequence has a limit in $\mathbb{R} \cup \{-\infty, +\infty\}$.