Question 1

Practice Midterm 2

First, we show $\liminf_{n\to\infty} s_n$ is an upper bound for A.
Fix $a \in A$. Then $\exists\ N_a$ s.t. $n \geq N_a$ ensures $s_n \geq a$. Thus, for $N > N_b$, $\inf\{s_n : n > N\} \geq a$.
By the contrapositive of HW4, Q6@, this gives

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} \geq a$$

Our argument above shows that $\liminf_{n\to\infty} s_n \geq \sup(A)$.
Suppose, for the sake of contradiction, that $\sup(A) < \liminf_{n\to\infty} s_n$. Then $\exists\ r \in \mathbb{R}$ s.t. $\sup(A) < r < \liminf_{n\to\infty} s_n$. Since $r \notin A$, $|\{n \in \mathbb{N} : s_n < r\}|$ is infinite. Thus, $\inf\{s_n : n > N\} \leq r\ \forall N \in \mathbb{N}$.

This implies $\liminf_{n\to\infty} s_n \leq r$, which is a contradiction.

(b) $\sup \emptyset = -\infty$

The reference to HW4 Q6(a) is now HW5 Q4(a).

Question 2

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(a) We prove the bounds by induction. The base case is $s_1 = 1$. If $1 \le s_n \le 2$, then

$$1 \le 1 + \frac{1}{2} \le s_{n+1} = 1 + \frac{1}{s_n} \le 2.$$

Thus $1 \le s_n \le 2$ for every $n \in \mathbb{N}$.

(b) We proceed by induction. For the base case, note that $s_1 = 1$, $s_2 = 2$, $s_3 = \frac{3}{2}$, and $s_4 = \frac{5}{3}$. Thus $s_3 \ge s_1$ and $s_4 \le s_2$.

Suppose $s_{2k+1} \ge s_{2k-1}$ for some $k \ge 1$. All terms are positive by part (a), so

$$s_{2k+2} = 1 + \frac{1}{s_{2k+1}} \le 1 + \frac{1}{s_{2k-1}} = s_{2k}.$$

Hence

$$s_{2k+3} = 1 + \frac{1}{s_{2k+2}} \ge 1 + \frac{1}{s_{2k}} = s_{2k+1}.$$

By induction, $s_{2k+2} \le s_{2k}$ and $s_{2k+1} \ge s_{2k-1}$ for all $k \ge 1$. Therefore the subsequence of even terms is decreasing and the subsequence of odd terms is increasing.

(c) Since $1 \le s_n \le 2$ for all n, the subsequences of even terms and odd terms are both bounded and monotone. Hence they both converge. Let

$$\lim_{k \to \infty} s_{2k} = s_{\text{even}}, \qquad \lim_{k \to \infty} s_{2k-1} = s_{\text{odd}}.$$

Note that:

$$s_{2k+1} = 1 + \frac{1}{s_{2k}} = 1 + \frac{1}{1 + \frac{1}{s_{2k-1}}}.$$

Since $s_{2k-1} \geq 1$, $s_{odd} \geq 1$. Thus, applying the limit theorems (quotient, sum), we have $s_{odd} = \lim_{k\to\infty} s_{2k+1} = 1 + \frac{1}{1 + \frac{1}{s_{odd}}}$.

Thus, s_{odd} solves $(s_{odd} - 1) = \left(1 + \frac{1}{s_{odd}}\right)^{-1}$

$\Leftrightarrow (s_{odd} - 1)\left(1 + \frac{1}{s_{odd}}\right) = 1 \Leftrightarrow s_{odd} + 1 - 1 + \frac{1}{s_{odd}} = 1$ ($s_{odd} \geq 1$)

$\Leftrightarrow s_{odd}^2 - s_{odd} - 1 = 0$. By the quadratic formula and the fact that $s_{odd} \in [1, 2]$, we obtain $s_{odd} = \varphi$.

Finally,

$$s_{2k} = 1 + \frac{1}{s_{2k-1}}.$$

Again, applying the limit theorems, we obtain $s_{even} = 1 + \frac{1}{s_{odd}} \Rightarrow s_{even} = \varphi$.

(d) Fix $\varepsilon > 0$. Choose N_{even} so $k > N_{even}$ ensures $|s_{2k} - \varphi| < \varepsilon$ and N_{odd} so $k > N_{odd}$ ensures $|s_{2k-1} - \varphi| < \varepsilon$. Let $N = 2 \cdot \max\{N_{even}, N_{odd}\}$. Then $n > N$ ensures that either $n = 2k$ and $k > N_{even}$ or $n = 2k - 1$ and $k > N_{odd}$. In either case $|s_n - \varphi| < \varepsilon$.

Question 3

Practice Midterm 2

(a) $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$

(b) If $\limsup_{n\to\infty} |s_n| < +\infty$, then $\lim_{N\to\infty} \sup\{|s_n| : n > N\} < +\infty$.

In particular, the sequence $a_N = \sup\{|s_n| : n > N\}$ is convergent, so $a_1 = \sup\{|s_n| : n > 1\} < +\infty$.

Thus $|s_n| \le \max\{a_1, |s_1|\}$ $\forall n \in \mathbb{N}$, so s_n is bounded.

(c) If s_n is bounded, $\exists\ M > 0$ s.t. $|s_n| < M$ $\forall n \in \mathbb{N}$. Thus $a_N = \sup\{|s_n| : n > N\} \le M$ for all $N \in \mathbb{N}$.

Hence $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} a_N \le M < +\infty$

(d) By part (b) s_n is bounded. By Bolzano Weierstrass, all bounded sequences have a convergent subsequence.

correction to (c): In the last line, the left-hand side should be $\limsup |s_n|$.

Question 4

Practice Midterm 2

(1) The first answer is **(a): limit exists**.

The second answer is **(b): decreasing**.

(2) (i) True.

(ii) False. Consider $s_n = (-1)^n$.