Homework 2 Solutions
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1(a) By $(M3)$, $1^2 = 1$, so by part (iv) of the theorem, $0 \cdot 1^2 = 1$. By $(M5)$, $0 \neq 1$, so $0 < 1$.

1(b) By part (v) of the Theorem, $0 < \frac{1}{b}$ and $0 < \frac{1}{a}$ so it suffices to show $\frac{1}{b} < \frac{1}{a}$.

Note that $a < b \Rightarrow a \leq b$

\[ a \cdot \left( \frac{1}{a} - \frac{1}{b} \right) = b \cdot \left( \frac{1}{a} - \frac{1}{b} \right) \Rightarrow a \cdot \left( \frac{1}{a} - \frac{1}{b} \right) \leq b \cdot \left( \frac{1}{a} - \frac{1}{b} \right) \]

Then

\[ \frac{1}{b} \cdot 1 \leq \frac{1}{a} \cdot 1 \Rightarrow \frac{1}{b} \leq \frac{1}{a} \]

It remains to show $\frac{1}{b} \neq \frac{1}{a}$. Suppose for the sake of contradiction that $\frac{1}{b} = \frac{1}{a}$.

Then

\[ \frac{1}{b} = \frac{1}{a} \Rightarrow \frac{1}{b} (ab) = \frac{1}{a} (ab) \]

which contradicts that $a < b$. Therefore $\frac{1}{b} \neq \frac{1}{a}$.

2 If $s_0 = \max(S)$, then $s_0 \in S$ and $s_0 \geq s \forall s \in S$. Consequently, $s_0$ is an upper bound of $S$.

Suppose $s_1$ is another upper bound of $S$. Since $s_0 \in S$, $s_1 \geq s_0$. Thus, $s_0$ is the least upper bound of $S$, so $s_0 = \sup(S)$. 

3) Mild notational difference: replace $a_i$ with $y_i$.

Base case: When $n=1$, $|a_1| \leq |a_1|$.

Inductive step: Assume $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + \ldots + |a_n|$.

By the triangle inequality (as stated in part 2) with $x = a_1 + a_2 + \ldots + a_n$ and $y = a_1 + \ldots + a_n$, we have $|a_1 + a_2 + \ldots + a_n| \leq |a_1| + \ldots + |a_n|$. By the inductive hypothesis, the right-hand side is bounded above by $|a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|$, which completes the proof.

4) By definition, $\sup(S)$ is $s$ for all $s \in S$. Thus, if $\sup(S) \notin S$, it is the largest element of $S$ and $\max(S) = \sup(S)$.

6) Suppose $S$ has a maximum $M_0$.

By Q2, $M_0 = \sup(S)$. This contradicts the fact that $\sup(S) \notin S$. Therefore, $S$ must not have a maximum.
5) Let's be an element of $S$. Since $\inf(S)$ is a lower bound for $S$, $\inf(S) \leq s$. Since $\sup(S)$ is an upper bound for $S$, $s \leq \sup(S)$. Therefore, $\inf(S) = \sup(S)$.

6) We will show $S = \{ \inf(s) \}^\sharp$, so there is one element in the set. Since $\inf(S) = \sup(S)$, $\inf(S)$ is both an upper and lower bound for $S$. In particular, for any $s \in S$, $\inf(S) \leq s$ and $\inf(S) \geq s$. Thus, $\inf(S) = s$ for all $s \in S$. This shows $S = \{ \inf(s) \}^\sharp$. 
Since inf(T) is a lower bound for the set T, if S \subseteq T, then inf(T) is also a lower bound for the set S. Since inf(S) is the greatest lower bound of S, inf(T) \leq inf(S). The fact that sup(S) \leq sup(T) follows from an analogous argument. The fact that inf(S) \leq sup(S) follows from Q5(a).

Since sup(s) is an upper bound for S and sup(T) is an upper bound for T, max{sup(S), sup(T)} is an upper bound for SUT. Thus, since sup(SUT) is the least upper bound for SUT, sup(SUT) \leq max{sup(S), sup(T)}.

By Q7(a), since S \subseteq SUT and T \subseteq SUT, we have sup(S) \leq sup(SUT) and sup(T) \leq sup(SUT). Thus max{sup(S), sup(T)} \leq sup(SUT).

Combining these two inequalities, we conclude max{sup(S), sup(T)} = sup(SUT).
(a) Since $S$ is bounded below, $\exists m_0 \in \mathbb{R}$ s.t. $s \geq m_0 \forall s \in S$. This implies $-m_0 \geq -s \forall s \in S$, so $-S$ is bounded above.

(b) Since $S$ is nonempty, so is $-S$. Since $-S$ is bounded above, by definition of the real numbers, it has a supremum, $\mathrm{sup}(-S)$.

(c) Since $\mathrm{sup}(-S)$ is an upper bound for $-S$, $-s \leq \mathrm{sup}(-S)$ for all $s \in S$, hence $s \geq -\mathrm{sup}(-S)$ for all $s \in S$. Therefore $-\mathrm{sup}(-S)$ is a lower bound for $S$, and it suffices to show it is the greatest lower bound.

For the sake of contradiction that $\mathrm{Sup}^*\mathcal{M}_0$ is a lower bound for $S$ with $\mathcal{M}_0 > -\mathrm{sup}(-S)$. As argued in part (a), $\mathcal{M}_0$ is an upper bound for $-S$. Furthermore, $\mathcal{M}_0 > -\mathrm{sup}(-S)$ implied $-\mathcal{M}_0 < \mathrm{sup}(-S)$. This is a contradiction since $\mathrm{sup}(-S)$ is the least upper bound for $-S$. 

Therefore \(-\sup(-S)\) must be the greatest lower bound of \(-S\).

Notational change: \(A+B\) becomes \(S+T\)

Step 1: Show that for all \(t \in T\), \(\inf(S+T) - t\) is a lower bound for \(S\).

By definition of \(S+T\) and the infimum, \(\inf(S+T)\) is a lower bound for \(S+T\), so
\[ s + t = \inf(S+T) \iff s \geq \inf(S+T) - t \quad \text{for all } s \in S, \ t \in T. \]
Thus, for all \(t \in T\), \(\inf(S+T) - t\) is a lower bound for \(S\).

Step 2: Show that \(\inf(S+T) - \inf(S)\) is a lower bound for \(T\).

By Step 1, for all \(t \in T\), \(\inf(S+T) - t\) is a lower bound for \(S\). By definition, \(\inf(S)\) is the greatest lower bound of \(S\). Thus,
\[ \inf(S) \leq \inf(S+T) - t \iff t \geq \inf(S+T) - \inf(S) \quad \text{for all } t \in T. \]
Since \( \inf(S+T) - \inf(S) \) is a lower bound for \( T \) and \( \inf(T) \) is the greatest lower bound,

\[
\inf(T) \geq \inf(S+T) - \inf(S).
\]

\[
\Rightarrow \quad \inf(S) + \inf(T) \geq \inf(S+T). \quad (*)
\]

It remains to prove the opposite inequality. Since \( \inf(S) \) and \( \inf(T) \) are lower bounds for \( S \) and \( T \), for all \( s \in S \) and \( t \in T \),

\[
\inf(S) \leq s \quad \text{and} \quad \inf(T) \leq t \Rightarrow \inf(S) + \inf(T) \leq s + t.
\]

Thus, \( \inf(S) + \inf(T) \) is a lower bound for \( S+T \). Since \( \inf(S+T) \) is the greatest lower bound,

\[
\inf(S) + \inf(T) \leq \inf(S+T). \quad (**)\]

Thus, combining inequalities (*) and (**) we obtain

\[
\inf(S) + \inf(T) = \inf(S+T). \quad \square
\]
slight notational change: $A = S, B = T$

9. Since $s \leq t$ for all $s \in S$ and $t \in T$, any $t \in T$ is an upper bound for $S$ and any $s \in S$ is a lower bound for $T$. Hence, $S$ is bounded above and $T$ is bounded below.

b) As shown in part(a), any $t \in T$ is an upper bound for $S$. Since $\sup(s)$ is the least upper bound, $\sup(s) \leq t$ for all $t \in T$. Thus, $\sup(s)$ is a lower bound for $T$, and since $\inf(t)$ is the greatest lower bound, $\sup(s) \leq \inf(t)$.

c) $S = [0,1], \ T = [1,2]$  
d) $S = [0,1), \ T = (1,2]$  

10. Throughout, we use $S$ to denote the set under consideration.

a) $\sup(s) = \sqrt{2}, \ \inf(s) = -\sqrt{2}$  
b) $\sup(s) = \pi, \ \inf(s) = -1$  
c) $\sup(s) = \inf(s) = 1$  
d) $S$ is not bounded above, $\inf(s) = 1$  
e) $\sup(s) = 1, \ \inf(s) = 0$
\( f \) \( \sup(S) = 1 \), \( \inf(S) = -1 \)
\( g \) \( S = [-1, 1] \), so \( \sup(S) = 1 \) and \( \inf(S) = -1 \)

11) (a) \( \sup(S) = 1 \), \( \inf(S) = 0 \)
(b) \( S \) is not bounded above, \( \inf(S) = 0 \)
(c) \( S \) is not bounded above, \( \inf(S) = 0 \)
(d) \( S \) is neither bounded above or below
(e) \( S = \{0, 1\} \), so \( \sup(S) = \inf(S) = 0 \)
(f) \( S \) is not bounded above, \( \inf(S) = 2^{\frac{1}{3}} \)
(g) \( \sup(S) = \inf(S) = 0 \)