Math 117: Homework 3
Due Friday, January 26th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*
Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded above. Prove that $a$ is the supremum of $S$ if and only if $a$ is an upper bound for $S$ and, for all $\epsilon > 0$, there exists $s \in S$ so that $s > a - \epsilon$.

Question 2
Consider $x, y \in \mathbb{R}$ satisfying $x, y \in [1, 2]$. Suppose $x^2 < 2$ and $y^2 > 2$.

(a) Suppose $0 < \epsilon < 1$. Prove that $(x + \epsilon)^2 \leq x^2 + 5\epsilon$ and $(y - \epsilon)^2 \geq y^2 - 4\epsilon$.

(b) Prove that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $x^2 + 5\epsilon_1 < 2$ and $y^2 - 4\epsilon_2 > 2$.

(c) Use parts (a) and (b) to show that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $(x + \epsilon_1)^2 < 2$ and $(y - \epsilon_2)^2 > 2$.

Question 3*
Consider an ordered field $F$. For any $a \in F$, recall that $a^2$ is an abbreviation for $a \cdot a$.

Fix $a \in F$ with $a \geq 0$. Consider the set $S = \{c \in F : c \geq 0, c^2 \leq a\}$

(a) Prove that $S$ is nonempty and bounded above.

(b) Suppose $F = \mathbb{R}$. Explain why, in this case, the supremum of $S$ exists.

Question 4*
Consider the set $S = \{c \in \mathbb{R} : c \geq 0, c^2 \leq 2\}$. Let $b = \sup(S)$.

(a) Prove that $b \in [1, 2]$.

(b) Prove that $b^2 \geq 2$. (Hint: proceed by contradiction, using question 2).

(c) Prove that $b^2 \leq 2$. (Hint: proceed by contradiction, using question 2).

Combining parts (b) and (c), we see that $b^2 = 2$.

In this way, we have shown there exists a real number $b > 0$ so that $b^2 = 2$. We can now define the symbol $\sqrt{2}$ by setting $\sqrt{2} := b$. In this way, we have proved $\sqrt{2} \in \mathbb{R}$. Combining this with our result from class that $\sqrt{2} \not\in \mathbb{Q}$, we see that $\sqrt{2} \in \mathbb{I}$, where $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.
Question 5*

(a) Prove the following, using the definition of convergence:

\[
\lim_{{n \to +\infty}} a^n = \begin{cases} 
0 & \text{if } |a| < 1, \\
1 & \text{if } a = 1.
\end{cases}
\]

(Hint: recall that the natural logarithm \( \log(x) \) is an increasing function for \( x > 0 \): that is, for any \( x, y > 0 \), \( x \leq y \iff \log(x) \leq \log(y) \).)

(b) If \( a \leq -1 \) prove that the sequence does not converge.

Question 6

Fix \( a \in \mathbb{R} \) and consider the collection of rational numbers \( S = \{ q \in \mathbb{Q} : a \leq q \} \).

(a) Suppose the underlying field is either \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{F} = \mathbb{Q} \). For which values of \( a \) does the minimum of \( S \) exist? Justify your answer with a proof.

(b) Suppose the underlying field is \( \mathbb{F} = \mathbb{R} \). Prove that \( \inf(S) = a \).

Question 7

Given \( s, t \in \mathbb{R} \), consider the set \( (s, t] = \{ x \in \mathbb{R} : s < x \leq t \} \). Find the maximum, minimum, supremum, and infimum of the set or state that they do not exist. Justify your answers with proofs.

Question 8

Prove that if \( s > 0 \), then there exists \( n \in \mathbb{N} \) satisfying \( \frac{1}{n} < s < n \).

Question 9*

Let \( a, b \in \mathbb{R} \). Show if \( a \leq b + \frac{1}{n} \) for all \( n \in \mathbb{N} \), then \( a \leq b \). (Compare to Question 3 on HW 1.)

Question 10*

(a) State the definition of what it means for a sequence \( s_n \) to converge to a limit \( s \)

(b) State the definition of what it means for a sequence \( s_n \) to not converge to a limit \( s \)

(c) Use the definition of a convergent sequence to prove that \( \lim_{{n \to +\infty}} \frac{n-3}{n^2+9} = 0 \).

(d) Use the definition of a convergent sequence to prove that the sequence \( s_n = (n + 1)^2 - 2 \) does not converge.
Question 11

Determine if the following sequences converge. Justify your answer with a proof.

(a) \( a_n = \frac{7n-19}{3n+7} \)

(b) \( b_n = \sin \left( \frac{n\pi}{3} \right) \)