Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

**Question 1**

Consider the sequences defined as follows:

\[ a_n = (-1)^{n+1}, \quad b_n = -\frac{1}{n}, \quad c_n = 2n. \]

(a) For each sequence, give an example of a monotone subsequence.

(b) For each sequence, give its set of subsequential limits. Justify your answer.

(c) For each sequence, give its lim inf and lim sup. Justify your answer.

(d) Which of the sequences converges? Diverges to \(+\infty\)? Diverges to \(-\infty\)? (You do not need to justify your answer.)

(e) Which of the sequences is bounded? (You do not need to justify your answer.)

**Question 2**

(a) State the definition of convergence for a sequence \( s_n \) to a limit \( s \).

(b) State what it means for a sequence \( s_n \) to not converge to a limit \( s \) by negating the definition from part (a).

(c) Suppose that \( s_n \) does not converge to \( s \in \mathbb{R} \). Prove that there exists \( \epsilon > 0 \) and a subsequence \( s_{n_k} \) so that \( |s_{n_k} - s| \geq \epsilon \) for all \( k \).

**Question 3**

One can show that the set of rational numbers \( \mathbb{Q} \) can be listed as a sequence \( r_n \). The exact procedure is a little tedious, but you can get an idea of how it works by considering the below diagram from the textbook. For example, \( r_1 = 0, r_2 = 1, r_3 = 1/2, \) and so on. Note that some numbers, such as \(-1\), are included multiples times.

(a) For any \( \epsilon > 0 \) and \( a \in \mathbb{R} \), show that the set \( \{ r \in \mathbb{Q} : |r - a| < \epsilon \} \) contains infinitely many elements. (**Hint:** Use denseness of the rationals.)

(b) Let \( r_n \) be the sequence of rational numbers. Use part (a) to show that for any \( a \in \mathbb{R} \), there exists a subsequence \( r_{n_k} \) that converges to \( a \). (**Hint:** Use part (a) to show that the set \( \{ n \in \mathbb{N} : |r_n - a| < \epsilon \} \) is infinite.)

(c) Let \( r_n \) be the sequence of rational numbers. Show that there exists a subsequence \( r_{n_k} \) satisfying \( \lim_{k \to +\infty} r_{n_k} = +\infty \).
Background on Infinite Series

In calculus, you encountered infinite series of the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$$

In fact, these are just limits of sequences. In particular, if we define the sequence

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

to be the sum of the first \( n \) terms of the series, then we define the value of the infinite series to be

$$\sum_{k=1}^{\infty} a_k = \lim_{n \to +\infty} s_n.$$

**DEFINITION 1.** Given a series \( \sum_{k=1}^{\infty} a_k \), define the sequence \( s_n = \sum_{k=1}^{n} a_k \). Then the series \( \sum_{k=1}^{\infty} a_k \) converges to a number \( L \) if and only if the sequence \( s_n \) converges to \( L \). Likewise, the series diverges to \( +\infty \) or \( -\infty \) if and only if the sequence \( s_n \) diverges to \( +\infty \) or \( -\infty \).

**Question 4* (Cauchy criterion)**

Recall that a sequence \( s_n \) is a Cauchy sequence if

for all \( \epsilon > 0 \), there exists \( N \in \mathbb{R} \) so that \( n, m > N \) ensures \( |s_n - s_m| < \epsilon \).

(a) Prove that the following is an equivalent definition of a Cauchy sequence:

\( s_n \) is a Cauchy sequence if, for all \( \epsilon > 0 \), there exists \( N \in \mathbb{R} \) so that \( n > m > N \) ensures \( |s_n - s_m| < \epsilon \).

(b) Prove the following theorem about series, known as the Cauchy criterion.

**THEOREM 1** (Cauchy Criterion). A series \( \sum_{k=1}^{\infty} a_k \) is convergent if and only if

for all \( \epsilon > 0 \) there exists \( N \in \mathbb{R} \) so that \( n > m > N \) ensures \( \left| \sum_{k=m+1}^{n} a_k \right| < \epsilon \).
(c) Now use Theorem 1 to prove the following corollary:

**COROLLARY 2.** If a series \( \sum_{k=1}^{\infty} a_k \) is convergent, then \( \lim_{k \to +\infty} a_k = 0 \).

(Hint: take \( n = m + 1 \) in the theorem from part (a).)

**Question 5**

(a) Prove the following by induction: for \( a \neq 1 \),
\[
\sum_{i=0}^{m-1} a^i = 1 + a + a^2 + \cdots + a^{m-1} = \frac{1 - a^m}{1 - a}.
\]

(b) Use part (a) to show that
\[
\sum_{i=n}^{m-1} a^i = a^n + a^{n+1} + \cdots + a^{m-2} + a^{m-1} = \frac{a^n - a^m}{1 - a}.
\]

(Hint: \( \sum_{i=n}^{m} a^i = \sum_{i=0}^{m-1} a^i - \sum_{i=0}^{n-1} a^i \).)

(c) Recall from HW2 Q3 that
\[
\left| \sum_{i=1}^{n} a_i \right| = |a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=1}^{n} |a_i|.
\]

Let \( s_n \) be a sequence such that \( |s_{n+1} - s_n| \leq 4^{-n} \) for all \( n \in \mathbb{N} \). Use part (b) and the above inequality to prove \( s_n \) is a Cauchy sequence.

(d) Does the sequence from part (c) converge? Justify your answer.

**Question 6* (decimal expansions)**

In this problem you will show that any number that can be represented as a nonnegative decimal expansion can be thought of as the limit of a bounded increasing sequence of real numbers. Since all bounded monotone sequences converge, this guarantees that any decimal expansion you can imagine represents (converges to) a real number.

Suppose we are given a decimal expansion \( K.d_1d_2d_3d_4 \ldots \), where \( K \) is a nonnegative integer and each \( d_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Let
\[
s_n = K + \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.
\]

(a) Show \( s_n \) is an increasing sequence. (This is almost obvious. Your proof should be short.)

(b) Use the result from Q5(a) to prove that \( \frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n} \).

(c) Use part (b) to prove that \( s_n \) is a bounded sequence.

(d) Since 0.\overline{9} = 0.999\ldots and 1 are both decimal expansions, by what you have shown, they both correspond to a real number. Use the hint from part (b) to show that they actually correspond to the same real number.
Question 7 (geometric series)

On previous homework/practice quizzes you proved the following results:

$$\lim_{n \to +\infty} r^n = \begin{cases} 
0 & \text{if } |r| < 1 \\
1 & \text{if } |r| = 1 \\
+\infty & \text{if } r > 1 \\
does \text{ not exist} & \text{if } r \leq -1,
\end{cases}$$

and

$$\sum_{k=1}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} \quad \text{for } r \neq 1.$$

(a) Prove that for $|r| < 1$, $\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$.

(b) Prove that for $|r| \geq 1$, $\sum_{k=1}^{\infty} r^k$ does not converge. (Hint: Use Corollary 2 from Q4.)

Question 8*

Let $s_n$ be a sequence of nonnegative numbers, and for each $n$ define $\sigma_n = \frac{1}{n} (s_1 + s_2 + \cdots + s_n)$.

(a) Show $\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n$.

(Hint: For the first inequality, show that $M > N$ implies

$$\inf \{\sigma_n : n > M\} \geq \left(1 - \frac{N}{M}\right) \inf \{s_n : n > N\}.$$  

For the last inequality, show first that $M > N$ implies

$$\sup \{\sigma_n : n > M\} \leq \frac{1}{M} (s_1 + s_2 + \cdots + s_N) + \sup \{s_n : n > N\}.$$

(b) Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

(c) Give an example for which $\lim \sigma_n$ exists but $\lim s_n$ does not exist.

Question 9

Suppose $s_n$ and $t_n$ are bounded sequences.

(a) Prove that $\limsup s_n + t_n \leq \limsup s_n + \limsup t_n$.

(b) Give an examples of bounded sequences $s_n$ and $t_n$ for which $\limsup s_n + t_n < \limsup s_n + \limsup t_n$. 

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