Homework 7 Solutions
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1) \( t \) is a subsequential limit of \( S_n \)
   \( \Downarrow \)
   there exists a subsequence \( S_{nk} \) of \( S_n \) s.t.
   \[ \lim_{k \to \infty} S_{nk} = t \]
   \( \downarrow \)
   if \( t \) is a real number, this follows since limit of product is product of limit. If \( t = \pm \infty \), this follows from result from class.
   there exists a subsequence \( S_{nk} \) of \( S_n \) s.t.
   \[ \lim_{k \to \infty} S_{nk} = -t \]
   \( \uparrow \)
   there exists a subsequence \( t_{nk} \) of \(-S_n\) s.t.
   \[ \lim_{k \to \infty} t_{nk} = t \]
   \( \downarrow \)
   \( t \) is a subsequential limit of \( S_n \)

2) First, suppose \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = s \in \mathbb{R} \).
   Fix \( \varepsilon > 0 \). There exists \( N, N_c \) s.t. \( n > N \) ensures \( |a_n - s| < \varepsilon \) and \( n > N_c \) ensures \( |c_n - s| < \varepsilon \).
   Furthermore, there exists \( N \) s.t. \( n > N \) ensures \( a_n \leq b_n \leq c_n \). Let \( \tilde{N} = \max \{ N, N_a, N_c, N_c \} \). Then \( n > \tilde{N} \) ensures \( s - \varepsilon < a_n \leq b_n \leq c_n < s + \varepsilon \), so \( |b_n - s| < \varepsilon \).
   This shows \( \lim_{n \to \infty} b_n = s \).
Next, suppose \( \lim_{n \to \infty} a_n = +\infty \). Fix \( M > 0 \).
There exists \( N \) s.t. \( n > N \) ensured \( a_n > M \).
There exists \( N \) s.t. \( n > N \) ensured \( a_n \leq b_n \).
Let \( \tilde{N} = \max \{ N, N' \} \). Then \( n > \tilde{N} \) ensured \( b_n > M \). This shows \( \lim_{n \to \infty} b_n = M \).

Finally, suppose \( \lim_{n \to \infty} c_n = -\infty \). Then
\(-c_n \leq -b_n \leq -a_n\) for all but finitely many \( n \) and \( \lim_{n \to \infty} -c_n = +\infty \). By the previous case, \( \lim_{n \to \infty} -b_n = +\infty \). Thus, \( \lim_{n \to \infty} b_n = -\infty \).

3a Claim: \( S = \{ \frac{1}{e} : e \in \mathbb{N} \} \)

Since \( s_{n_k} = \frac{1}{k} \) is a subsequence,
OES. Since \( s_{n_k} = \frac{1}{k} \) is a subsequence for all \( e \in \mathbb{N} \), \( \frac{1}{e} \in S \).

It remains to show no other real number or \( +\infty \) belongs to \( S \).

Neither \( +\infty \) nor \( -\infty \) belong to \( S \), since the sequence is bounded.

Suppose \( a \in S \) for some \( a \in \mathbb{R} \). By the main subsequence theorem, it suffices to
To show $\exists \varepsilon > 0$ so that $|a - s_n| \geq \varepsilon$ for all $n$.

If $a > 1$, then $|a - s_n| \geq |a - 1| = \varepsilon \forall n$.

If $a < 0$, then $|a - s_n| > |a| = \varepsilon \forall n$.

If $\frac{1}{2} > a > \frac{1}{2} + 1$ for some $k \in \mathbb{N}$, then $|a - s_n| \geq \min \{ |a - \frac{1}{2}|, |a - \frac{1}{2} + 1| \} = \varepsilon \forall n$.

This completes the proof.

(b) $\limsup s_n = \max (s) = 1$

$\liminf s_n = \min (s) = 0$
(a) $s_n$ is a bounded sequence if $\exists M > 0$ s.t. $|s_n| < M$ for all $n \in \mathbb{N}$.

(b) Assume for the sake of contradiction that $\exists k \in \mathbb{N}$ s.t. $B_k : \exists s_n : s_n > s - \frac{1}{k}$ has finitely many elements.

Case 1: $B_k$ has zero elements.
Then $s_n \leq s - \frac{1}{k}$ for all $n \in \mathbb{N}$.
This contradicts the fact that $s$ is the least upper bound.

Case 2: $B_k$ has a nonzero number of elements. Then $B_k$ has a maximum $M_k := \max B_k$. Since $s_n < s$ for all $n$, $M_k < s$. Also, note that if $s_n \notin B_k$, then $s_n \leq s - \frac{1}{k} = M_k$. Thus $M_k$ is an upper bound for $\exists s_n : n \in \mathbb{N}$. Since $M_k < s$, this contradicts that $s$ was the least upper bound.

Therefore, $B_k$ has infinitely many elements for all $k \in \mathbb{N}$. 
(c) Fix $\varepsilon > 0$. Choose $k \in \mathbb{N}$ so that $\frac{1}{k} < \varepsilon$. Then $\exists n : |s_n - s| < \varepsilon^2 \implies \exists n : |s_n - s| < \frac{1}{k^2}$

$$= \{n : s - \frac{1}{k} < s_n < s + \frac{1}{k}\}$$

since $s_n < s \forall n \implies \exists n : s - \frac{1}{k} < s_n < s + \frac{1}{k}$

Furthermore $\exists n : s - \frac{1}{k} < s_n < s + \frac{1}{k}$, since each element $s_n$ in $B_k$ corresponds to at least one index $n$ in $\exists n : s - \frac{1}{k} < s_n < s + \frac{1}{k}$.

By part (b), we obtain $|\exists n : |s_n - s| < \varepsilon^2| = +\infty$. Thus, by the main subsequences theorem, there is a subsequence of $s_n$ converging to $s$.

(d) Define $s_n = \frac{1}{n}$. Then $s = \sup \{s_n : n \in \mathbb{N}\} = 1$, but since $\lim_{n \to \infty} s_n = 0$, all subsequences of $s_n$ converge to $0$. 
5. (a) If \( s_{nk} \) is bounded, by Bolzano-Weierstrass, \( s_{nk} \) must have a convergent subsequence \( s_{nke} \). Since \( s_{nke} \) is also a subsequence of \( s_n \), \( s_n \) has a convergent subsequence.

(b) Suppose \( |s_n| \) does not diverge to \( +\infty \). Then \( \exists M > 0 \) s.t. \( \forall N, \exists n > N \) for which \( |s_n| \leq M \). Since \( |s_n| = 0 \) for all \( n \in \mathbb{N} \), this implies there exist infinitely many \( n \in \mathbb{N} \) for which \( 0 \leq |s_n| \leq M \). Consequently, there exists a subsequence \( s_{nke} \) for which \( 0 \leq |s_{nke}| \leq M \) \( \forall k \in \mathbb{N} \). Therefore \( s_{nk} \) is a bounded sequence, so by part (a), \( s_n \) must have a convergent subsequence.

6. (a) If \( \lim s_n = s \), then all subsequences of \( s_n \) also converge to \( s \). Hence every subsequence \( s_{nk} \) has a further subsequence \( s_{nke} = s_{nk} \) that converges to \( s \).
Suppose \( \lim s_n \neq s \). Then, \\
\[ \exists \varepsilon > 0 \text{ s.t. } \forall N, \exists n > N \text{ s.t. } |s_n - s| \geq \varepsilon \]

First, taking \( N = 1 \), we have \( \exists n_1 > 1 \text{ s.t. } |s_{n_1} - s| \geq \varepsilon \). Suppose we have chosen \( n_k \). Taking \( N = n_{k-1} \), we see that \( \exists n_k > n_{k-1} \text{ s.t. } |s_{n_k} - s| \geq \varepsilon \).

Therefore there exists a subsequence \( s_{n_k} \) s.t. \( |s_{n_k} - s| \geq \varepsilon \) \( \forall k \). Since \( s_{n_k} \) is always at least distance \( \varepsilon \) from \( s \), no further subsequence of \( s_{n_k} \) can converge to \( s \).

If \( \sum_{k=1}^{\infty} a_k = +\infty \), then \( s_n := \sum_{k=1}^{n} a_k \) diverges to \( +\infty \).

Since \( 0 \leq a_k \leq b_k \), \( t_n := \sum_{k=1}^{n} b_k \geq s_n \). The result then follows from the generalized squeeze lemma.
Define $S_n = \sum_{k=1}^{n} a_k$, $T_n = \sum_{k=1}^{n} b_k$.

Note that our hypotheses ensure $S_n$ converges to $A \in \mathbb{R}$ and $T_n$ converges to $B \in \mathbb{R}$.

\[ \lim_{n \to \infty} (S_n + T_n) = \lim_{n \to \infty} S_n + \lim_{n \to \infty} T_n = A + B \]

Suppose $\sum_{k=1}^{\infty} |a_k|$ is convergent, so it satisfies the Cauchy criterion. We will show $\sum_{k=1}^{\infty} |a_k|$ converges by showing it satisfies the Cauchy criterion.

Fix $\varepsilon > 0$. \( \exists \ N \ s.t. \ n > m > N \) implies
\[ \left| \sum_{k=m+1}^{n} |a_k| \right| < \varepsilon. \]

Since \( \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| \), we have

that \( n > m > N \) implies

\[ \left| \sum_{k=m+1}^{n} a_k \right| < \varepsilon. \]

Thus, \( \sum_{k=1}^{\infty} a_k \) satisfies the Cauchy criterion.