Def (Cauchy sequence): A sequence \( s_n \) is a Cauchy sequence if for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{R} \) s.t. \( m,n > N \) ensures \( |s_n - s_m| < \varepsilon \)

How do Cauchy sequences fit in with the types of sequences we already know?

Lemma: Convergent sequences are Cauchy sequences

Pf: Suppose \( s_n \) is a convergent sequence, that is \( \lim_{n \to \infty} s_n = s \), for \( s \in \mathbb{R} \). Fix \( \varepsilon > 0 \). Since \( \lim_{n \to \infty} s_n = s \), \( \exists N \) s.t. \( n > N \) ensured \( |s_n - s| < \frac{\varepsilon}{2} \).
Thus, for \( m,n > N \), we have

\[
|s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
Since \( \varepsilon > 0 \) was arbitrary, \( s_n \) is Cauchy. \( \square \)

**Lemma:** Cauchy sequences are bounded.

The proof is similar to the proof that convergent sequences are bounded.

**Remark:** Recall the reverse triangle inequality: for \( a, b \in \mathbb{R} \),
\[ |a - b| \leq |a - c| + |c - b|. \]
Since \( x \leq |x| \) \( \forall x \in \mathbb{R} \), we have \( |a - b| \leq |a - b| \).

**Pf:** Let \( \varepsilon = 2 \). Since \( s_n \) is Cauchy, \( \exists N \) s.t.
\[ m, n > N \] ensures \( |s_n - s_m| < 2 \), which implies by reverse triangle inequality
\[ |s_n - s_m| < 2 \iff |s_n| < |s_m| + 2. \]

In particular, if \( n > N \), we have \([N] + 1 > N\), so
\[ |s_n| < 2 + |s_{[N]+1}| \]. Define
\[ M = \max \{ 2 + |s_{[N]+1}|, |s_1|, |s_2|, ..., |s_N| \} \].
Then we have \( |s_n| < M \) for all \( n \geq N \).
Thus, \( s_n \) is a bounded sequence. \( \square \)
MAJOR THEOREM #4

Thm: A sequence is convergent iff it is Cauchy.

Here you must know what the limit is and show elts of sequence get close to it.

Here you must know that elts of sequence “bunch up.” (Don’t need to know what they are bunching up around.)

Remark:
- If $s_n \leq b$ for all but finitely many $n$ and the limit of $s_n$ exists, then $\lim s_n \leq b$.
- If $a \leq b + \varepsilon$ for all $\varepsilon > 0$, then $a \leq b$.

Pf:
- We already proved that convergent sequences are Cauchy sequences, so it remains to show that Cauchy sequences converge.
- Suppose $s_n$ is Cauchy. By theorem from last time, it suffices to show $\lim_{n \to \infty} s_n = \limsup_{n \to \infty} s_n$ to conclude that $\lim_{n \to \infty} s_n$ exists. Since we already showed Cauchy sequences are bounded,
it would be impossible for \( \lim_{n \to \infty} s_n \) to equal \(+\infty\) or \(-\infty\). Thus, the sequence must converge.

- **Fix \( \varepsilon > 0 \).** Since \( s_n \) is Cauchy, \( \exists N \) s.t. \( n, m > N \) ensures \( |s_n - s_m| < \varepsilon \) \( \iff \) \( s_m - \varepsilon < s_n < s_m + \varepsilon \).

Thus, for \( m > N \), we have
\[
a_N = \sup \{ s_{n}: n > N | \exists s_{m} \leq s_{m} + \varepsilon \iff a_N - \varepsilon \leq s_{m}.
\]

Thus, for \( m > N \), we have
\[
a_N - \varepsilon \leq \inf \{ s_{m}: m > N | \exists s_{m} \leq b_{N} \} = b_{N}.
\]

Since \( a_n \) is a decreasing sequence and \( b_n \) is an increasing sequence, for all \( k > N \),
\[
ak - \varepsilon \leq a_N - \varepsilon \leq b_N \leq b_k.
\]

By Remark,
\[
\limsup_{n \to \infty} s_n - \varepsilon = \lim_{k \to \infty} a_k - \varepsilon = b_N \leq \lim_{k \to \infty} b_k = \lim_{n \to \infty} s_n.
\]

Thus, \( \limsup_{n \to \infty} s_n \leq \lim_{n \to \infty} s_n + \varepsilon \).
Since $\epsilon > 0$ was arbitrary, by remark,
\[ \limsup_{n \to \infty} s_n \leq \liminf_{n \to \infty} s_n. \]

We always have \( \limsup_{n \to \infty} s_n \leq \liminf_{n \to \infty} s_n \). Thus
\[ \liminf_{n \to \infty} s_n = \limsup_{n \to \infty} s_n. \]

== Types of Sequences ==

<table>
<thead>
<tr>
<th>BOUNDED</th>
<th>MONOTONE</th>
<th>NOT MONOTONE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_n = \frac{1}{n}$</td>
<td>$s_n = \frac{(-1)^n}{n}$</td>
<td>$s_n = (-1)^n$</td>
</tr>
<tr>
<td>CAUCHY SEQUENCES</td>
<td>CONVERGENT SEQUENCE</td>
<td></td>
</tr>
<tr>
<td>CONVERGENT SEQUENCE</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_n = n^3$</td>
<td>$s_n = \begin{cases} (-1)^n, &amp; n \leq 4 \ \frac{1}{n}, &amp; n &gt; 4 \end{cases}$</td>
<td>$s_n = (-1)^n$</td>
</tr>
<tr>
<td>DIVERGE TO $+\infty$ OR $-\infty$</td>
<td></td>
<td>THE LIMIT EXISTS</td>
</tr>
</tbody>
</table>