Goal: we know a lot about monotone sequences... what can we say about bounded sequences.

First, recall...

**Def (sequence):** A sequence is a function whose domain is a set of the form \( \{m, m+1, m+2, \ldots\} \) for some \( m \in \mathbb{Z} \). We study sequences whose range is \( \mathbb{R} \).

**Remark:** While we could write \( s(n) \), we use \( s_n \) to emphasize that sequences are a
special type of functions!

Now, we will define the notion of subsequence.

**Def (subsequence):** Consider a sequence $s_n$. For any sequence $n_k$ of natural numbers satisfying $n_1 < n_2 < n_3 < \ldots$, a sequence of the form $s_{n_k}$ is a subsequence of $s_n$.

**Remark:** We could write $s_n$ as $s(n)$, $n_k$ as $n(k)$, and $s_{n_k}$ as $s(n(k))$.

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

**Ex 1:** $s_n = (-1, 2, -3, 4, \ldots, (-1)^n n, \ldots)$

$$s_{n_k} = (-1, -3, -5, \ldots, (-1)^{(2k-1)} (2k-1), \ldots)$$

$$n_k = (1, 3, 5, \ldots, 2k-1, \ldots)$$
Note that
\[
\begin{align*}
    a_N &= \sup \{ s_n : n > N^2 \} = (\infty, \infty, \ldots, \infty, \ldots) \\
    b_N &= \inf \{ s_n : n > N^2 \} = (0, 0, \ldots, 0, \ldots)
\end{align*}
\]

\text{Ex 6: } s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, \ldots, n, \ldots)

\text{Ex 7: } s_n = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, (2k)^{-1}, \ldots)

\text{Ex 8: } s_n = (2, 4, 6, \ldots, 2k, \ldots)

\text{Ex 9: } s_n = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, (2k)^{-1}, \ldots)
**Limits of Subsequences**

**Lemma:** Given a sequence $s_n$, $n \in \mathbb{N}$, if $s_{nk}$ is a subsequence, then $n_k \geq k$ for all $k \in \mathbb{N}$.

**Pf:** Base case: When $k=1$, $n_1 \geq 1$ since $n_k \in \mathbb{N}$ for all $k$.

Inductive step: Assume $n_{k-1} \geq k-1$. Since $n_k > n_{k-1}$, we have $n_k \geq n_{k-1} + 1 \geq k$.

**Def:** (subsequential limit) A **subsequential limit** of a sequence $s_n$ is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of $s_n$.

**Ex:** $s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots)$

0 and $+\infty$ are subsequential limits.

**Thm:** If a sequence $s_n$ converges to a limit $s$, then every subsequence also converges to $s$. 
Let $s_{n_k}$ be an arbitrary subsequence of $s_n$. Fix $\varepsilon > 0$. Since $\lim_{n \to \infty} s_n = s$, there exists $N$ such that $n > N$ ensures $|s_n - s| < \varepsilon$. If $k > N$, then $n_k = k > N$, so $|s_{n_k} - s| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\lim_{k \to \infty} s_{n_k} = s$.

**Ex:** $s_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$
- $\emptyset$ is the set of all subsequential limits

**Thm (main subsequences theorem):**
Let $s_n$ be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$
- The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$
  - if and only if
- $t$ is a subsequential limit of $s_n$.

(b) $s_n$ is unbounded above $\iff +\infty$ is a subseq. limit.
(c) $s_n$ is unbounded below $\iff -\infty$ is a subseq. limit.

**Mental image (a):**

```
\begin{tikzpicture}
    \draw[->] (0,0) -- (7,0) node[right] {$n$};
    \draw[->] (0,0) -- (0,5) node[above] {$s_n$};
    \draw (0,1.5) -- (5,1.5) node[above] {$t$};
    \draw[dashed] (0,0) -- (5,0);
    \draw[dashed] (0,1.5) -- (5,1.5);
    \draw (1,0.5) -- (1,0) -- (5,0);
    \draw (2,1) -- (2,0) -- (5,0);
    \draw (3,0.5) -- (3,0) -- (5,0);
    \draw (4,1) -- (4,0) -- (5,0);
    \draw (5,0) -- (5,0.5) node[right] {$\varepsilon / 2$};
    \draw (5,0) -- (5,1);\end{tikzpicture}
```
Lemma: If $s_n$ is unbounded above, the set $\{n: s_n > M\}$ is infinite for all $M > 0$.

Proof: Assume, for the sake of contradiction, that there exists $M > 0$ for which $\{n: s_n > M\}$ is finite. Define

$$s_{\text{max}} = \max \{s_n: s_n > M\}.$$

Then define $\tilde{M} = \max \{s_{\text{max}}, M\}$

- if $s_n > M$, $s_n \leq s_{\text{max}} \leq \tilde{M}$
- if $s_n \leq M$, $s_n \leq \tilde{M}$.

Thus, for all $n \in \mathbb{N}$, $s_n \leq \tilde{M}$, so $s_n$ is bounded above, which is a contradiction. $\square$
Proof of Main Subsequences Theorem

(a) Suppose \( \exists n: |s_n - t| < \varepsilon \) is infinite for all \( \varepsilon > 0 \).

We can construct a subsequence of \( s_n \) in the following way:

Choose \( s_{n_1} \) so that \( |s_{n_1} - t| < 1 \).
Choose \( s_{n_2} \) so that \( |s_{n_2} - t| < \frac{1}{2} \) and \( n_2 > n_1 \).

Choose \( s_{n_k} \) so that \( |s_{n_k} - t| < \frac{1}{k} \) and \( n_k > n_{k-1} \).

Note that \( |s_{n_k} - t| < \frac{1}{k} \Leftrightarrow t - \frac{1}{k} < s_{n_k} < t + \frac{1}{k} \) for all \( k \in \mathbb{N} \). So by the squeeze lemma, \( \lim_{k \to \infty} s_{n_k} = t \), so \( \lim_{k \to \infty} s_{n_k} = t \) and \( t \) is a subsequential limit.

Now, suppose \( t \) is a subsequential limit of \( s_n \).

Fix \( \varepsilon > 0 \). Since there exists a subsequence \( s_{n_k} \) that converges to \( t \), there exists \( N \) s.t. \( k > N \) ensures \( |s_{n_k} - t| < \varepsilon \).

Therefore, \( \{ s_{n_k}: k > N \} \subseteq \{ s_n: |s_n - t| < \varepsilon \} \).

Since \( \{ s_{n_k}: k > N \} \) is infinite, so is \( \{ s_n: |s_n - t| < \varepsilon \} \).
(b) Suppose \([s_n \text{ is unbounded above}]\). By the lemma, for all \(M > 0\), \(\exists n: s_n \geq M\) is infinite. Hence, we may construct a subsequence as follows.

Choose \(n_1\) so that \(s_{n_1} > 1\).

Choose \(n_2\) so that \(s_{n_2} > 2\) and \(n_2 > n_1\).

Choose \(n_k\) so that \(s_{n_k} > k\) and \(n_k > n_{k-1}\).

Fix \(\tilde{M} > 0\). For \(k > \tilde{M}\), \(s_{n_k} > k > \tilde{M}\).

Since \(\tilde{M}\) was arbitrary, \(\lim_{k \to \infty} s_{n_k} = +\infty\).

Thus \(+\infty\) is a subsequential limit.

Suppose \([+\infty \text{ is a subsequential limit}]\). Assume, for the sake of contradiction, that \(s_n\) is bounded above, that is there exists \(M > 0\) s.t. \(s_n \leq M\) for all \(n \in \mathbb{N}\). Take \(s_{n_k}\) s.t. \(\lim_{k \to \infty} s_{n_k} = +\infty\).

Then \(s_{n_k} \leq M\) for all \(k \in \mathbb{N}\). This is a contradiction.

(c) Note that \([s_n \text{ is unbounded below}]\)
$-\infty$ is unbounded above]

(b)

$+\infty$ is a subsequential limit of $-\infty$

$c$

$-\infty$ is a subsequential limit of $\infty$