Thm (main subsequences theorem)
Let $s_n$ be a sequence of real numbers.
(a) Let $t \in \mathbb{R}$
   \[
   \text{The set } \{n : |s_n - t| < \varepsilon\} \text{ is infinite for all } \varepsilon > 0
   \]
   if and only if
   \[
   \{t \text{ is a subsequential limit of } s_n\}
   \]
(b) $s_n$ is unbounded above $\iff +\infty$ is a subseq. limit.
(c) $s_n$ is unbounded below $\iff -\infty$ is a subseq. limit.

Why are subsequences important?

Even though not all sequences are monotone

Thm: Every sequence $s_n$ has a monotonic subsequence.
Proof: We will say that the $n$th element of a sequence is dominant if it is greater than every element that follows, that is $S_n$ is dominant if $S_n > S_m$ for all $m > n$. 

\[\text{Case 1:} \text{ Suppose } S_n \text{ has infinitely many dominant elements.}\]

Define $S_{n_k}$ to be the subsequence of dominant terms. Then $S_{n_k} > S_{n_{k+1}}$ for all $k \in \mathbb{N}$, so $S_{n_k}$ is decreasing, hence monotone.

\[\text{Case 2:} \text{ Suppose } S_n \text{ has finitely many dominant elements.}\]

- Choose $n_1$ so that $S_{n_1}$ is beyond all of the dominant elements in the sequence.
- Since $S_{n_1}$ is not dominant, there exists $n_2 > n_1$ so that $S_{n_2} \geq S_{n_1}$.
- Since $S_{n_k}$ is not dominant, there exists $n_{k+1} > n_k$ so that $S_{n_{k+1}} \geq S_{n_k}$.

Thus we have found a subsequence that is increasing, hence monotone. \(\square\)
**MAJOR THEOREM 5**

**Thm (Bolzano-Weierstrass):** Every bounded sequence has a convergent subsequence.

**Pf:** If \( s_n \) is a bounded sequence, the previous theorem ensures there exists a subsequence \( s_{n_k} \) that is monotonic (and also bounded). Since all bounded, monotone sequences converge, \( s_{n_k} \) is convergent.