Lecture 13
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How do subsequences relate to lim inf and lim sup?

Downside: in general $a_n, b_n$ are not subsequences of $s_n$.

Upside:

**Thm:** For any sequence $s_n$, $\limsup s_n$ and $\liminf s_n$ are subsequential limits.

**Pf:** First, we will show $\limsup s_n$ is a subsequential limit.

**CASE 1:** Suppose $\limsup s_n = -\infty$. Since $\liminf s_n \leq \limsup s_n$, then $\liminf s_n = -\infty$, so...
\[ \lim_{n \to \infty} s_n = -\infty. \]

**CASE 2:** Suppose \( \limsup_{n \to \infty} s_n = +\infty \), that is \( \lim_{n \to \infty} an = +\infty \). Fix arbitrary \( M > 0 \). Then there exists \( N_0 \) s.t. \( N > N_0 \) ensures \( an > M \). Thus \( M \) is not an upper bound of \( \{ sn : n > N \} \) when \( N > N_0 \), so there exists \( s_{N_1} > M \). Thus \( s_n \) is not bounded above. Hence \( \infty \) is a subsequential limit.

**CASE 3:** Suppose \( \lim_{n \to \infty} s_n = t \) for \( t \in \mathbb{R} \), that is \( \lim_{n \to \infty} an = t \). Fix arbitrary \( \varepsilon > 0 \). We will show \( \{n: t - \varepsilon < s_n < t + \varepsilon\} = \{n: |an-t| < \varepsilon\} \) is infinite.

By defn of convergence of \( an \) to \( t \), \( \exists N_0 \) s.t. \( N > N_0 \) ensures \( |an-t| < \varepsilon \). Then \( \sup\{sn : n > N_0\} = an < t + \varepsilon \). In particular, for \( N = N_0 + 1 \), \( \sup\{s_n : n > N_0 + 1\} < t + \varepsilon \). Thus for all \( n > N_0 + 1 \), \( s_n < t + \varepsilon \).
Suppose, for the sake of contradiction, that \( \{n: t-\varepsilon < s_n < t+\varepsilon\} \) is finite. Since we know \( n > N_0 + 1 \) ensures \( s_n < t + \varepsilon \), there must be \( N_1 > N_0 + 1 \) for which \( s_n \leq t - \varepsilon \) for all \( n > N_1 \).

Then \( a_n = \text{sup}\{s_n: n > N\} \leq t - \varepsilon \) for \( N > N_1 \). This implies \( \lim_{n \to \infty} a_n \leq t - \varepsilon \). This contradicts that \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} s_n = t \). Therefore, \( \{n: t-\varepsilon < s_n < t+\varepsilon\} \) is infinite. Since \( \varepsilon > 0 \) was arbitrary, by main subseq. theorem, \( t \) is a subsequential limit.

Next, we show \( \lim_{n \to \infty} s_n \) is a subsequential limit.

**Fact:** \( \lim_{n \to \infty} s_n = \limsup_{n \to \infty} -s_n \)

Thus, by what we've already shown, \( \lim_{n \to \infty} s_n \) is a subsequential limit of \( -s_n \).

**Fact:** \( t \) is a subseq. limit of \( s_n \) \iff \( -t \) is a subseq. limit of \( -s_n \).
Thus \( \lim_{n \to \infty} s_n \) is a subseq limit of \( s_n \).

In fact, \( \limsup_{n \to \infty} s_n \) and \( \lim_{n \to \infty} s_n \) aren't just any subsequential limit: they are the largest and smallest subsequential limit.

Recall: squeeze lemma

Given \( a_n \leq b_n \leq c_n \) for all \( n \in \mathbb{N} \), if

\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n,
\]

then \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n \).

Thm: Let \( S \) denote the set of subsequential limits of \( s_n \), then \( \limsup_{n} s_n = \max(S) \) and \( \liminf_{n} s_n = \min(S) \).

Pf: By the previous theorem, we have \( \limsup_{n \to \infty} s_n \in S \) and \( \lim_{n \to \infty} s_n \in S \), so it suffices to show that, for all \( t \in S \), we have \( \lim_{n \to \infty} s_n \leq t \leq \limsup_{n \to \infty} s_n \). Suppose \( \lim_{k \to \infty} s_{n_k} = t \).
Since \( \eta_k = k \), \( \{ s_{nk} : k > N \} \leq \{ s_{nim} : n > N \} \) for any \( N \in \mathbb{R} \). Thus

\[
\begin{align*}
b_N &= \inf \{ s_{nim} : n > N \} \leq \inf \{ s_{nk} : k > N \} \\
&\leq \sup \{ s_{nk} : k > N \} \leq \sup \{ s_{nim} : n > N \} = a_N
\end{align*}
\]

Sending \( N \to \infty \),

\[
\lim_{n \to \infty} s_n = \lim_{N \to \infty} b_N \leq \lim_{k \to \infty} s_{nk} = t = \limsup s_{nk} \leq \lim_{N \to \infty} a_N = \limsup s_{nim} \quad \square
\]