Lecture 4
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We'll study two major theorems for IR: Archimedean Property and Q is dense in IR.

MAJOR RESULT #1

Thm (Archimedean Property): If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$, then there exists $n \in \mathbb{N}$ so that $na > b$.

Remark: Even if $a$ is really small and $b$ is huge, some integer multiple of $a$ is bigger than $b$.

"Given enough time, one can empty a large bathtub with a small spoon."

We will prove by contradiction.

Scratchwork:

$P = [\forall a, b > 0, \exists n \in \mathbb{N} \text{ s.t. } na > b]$

$\neg P = [\exists a, b > 0 \text{ s.t. } \forall n \in \mathbb{N}, na \leq b]$
\textbf{Proof:} Assume, for the sake of contradiction, that \( \exists a, b \in \mathbb{R} \) with \( a > 0, b > 0 \) s.t. for all \( n \in \mathbb{N}, na \leq b. \)

Define \( S = \{ na : n \in \mathbb{N} \} \), so \( b \) is an upper bound for \( S \). Since \( S \) is a nonempty subset of \( \mathbb{R} \) that is bounded above, by the definition of \( \mathbb{R} \), \( S \) has a supremum. Define \( s_0 = \sup(S) \).

Since \( a > 0 \), we have \( s_0 - a < s_0 < s_0 + a \).

Since \( s_0 = \sup(S) \), there exists \( n_0 \in \mathbb{N} \) s.t. \( s_0 - a < n_0a \Rightarrow s_0 < (n_0 + 1)a \).

Since \((n_0 + 1)a \in S\), this contradicts the fact that \( s_0 \) is an upper bound of \( S \). \( \square \)
As a consequence of the Archimedean Property, we have a few useful lemmas...

**Lemma:** For any \( a \in \mathbb{R} \), there exists \( n \in \mathbb{N} \) s.t. \( a < n \).

\[
\text{If } a \leq 0 \text{ then the result holds for } n = 1. \\
\text{If } a > 0 \text{, then since } 1 > 0 \text{, by A.P. there exists } n \in \mathbb{N} \text{ s.t. } 1 \cdot n > a.
\]

\( \square \)

**Lemma:** For any \( a, b \in \mathbb{R} \), \( a < b \), there exists \( n \in \mathbb{N} \) so that \( a + \frac{1}{n} < b \).

Mental image:

\[
\begin{array}{c}
\downarrow \\
\frac{1}{n} \\
\hline
a \\
\frac{a + \frac{1}{n}}{b
\end{array}
\]

\[
\text{Let } y = b - a > 0 \text{ and } 1 > 0. \text{ By A.P., there exists } n \in \mathbb{N} \text{ s.t. } ny > 1 \Leftrightarrow y > \frac{1}{n} \\
\Leftrightarrow b - a > \frac{1}{n} \Leftrightarrow a + \frac{1}{n} < b.
\]

\( \square \)
Lemma: If $x, y \in \mathbb{R}$ satisfy $1 < x - y$, then $\exists m \in \mathbb{Z}$ so that $y < m < x$.  

Mental image: 

Proof: By the first lemma, there exists $n \in \mathbb{N}$ s.t. $n > y$. Define $S = \{ j \in \mathbb{Z} : y < j \leq n \}$. Then $S$ is nonempty and finite, so $m = \min(S)$ exists. By defn of $m$, $m \in \mathbb{Z}$, $y < m$, and $m - 1 \leq y$. Therefore, $y < m \leq 1 + y < x$. \qed
Now, we will apply the previous lemmas to prove...

**MAJOR THEOREM #2**

**Thm. (Q is dense in \( \mathbb{R} \)):** If \( a, b \in \mathbb{R} \) with \( a < b \), there exists \( r \in \mathbb{Q} \) satisfying \( a < r < b \).

Mental image: 

\[
\begin{array}{c}
\mathbb{Q} \rightarrow r \\
\mathbb{R}
\end{array}
\]

This is similar to the result we proved on the first day that between any two rational numbers there is a rational number.

**Pf:** By the lemma, \( \exists \ n \in \mathbb{N} \ s.t. \ a + \frac{1}{n} < b \) \( \iff \) \( na + 1 < bn \iff 1 < bn - an \). By the other lemma, there exists \( m \in \mathbb{Z} \) so that \( an < m < bn \iff a < \frac{m}{n} < b \). \( \square \)
We now have all the tools we need to rigorously prove our previous claims about the minimum/maximum/infimum/supremum of subsets of IR! For example...

Prop: For $a, b \in \text{IR}$, $a < b$, the set $S = [a, b)$ does not have a maximum and $\sup(S) = b$.

Proof:
First, we show that $S$ does not have a maximum. Assume, for the sake of contradiction, that $\max(S) = M_o$. Since $M_o \in S$, $a < M_o < b$. By density of IR in IR, $\exists r \in \text{IR}$ s.t. $M_o < r < b$, so $r \in S$. This contradicts that $M_o$ was the largest element in $S$.

Now, we show $\sup(S) = b$. By defn of $S$, $b$ is an upper bound. Suppose $M_o$ is another upper bound of $S$. If $M_o < b$, then by density of IR in IR, $\exists r \in \text{IR}$ s.t. $M_o < r < b$, so $r \in S$, which is a contradiction. Thus, $M_o \geq b$, so $b$ is the least upper bound. \(\square\)
Going forward, we will use $+\infty$ and $-\infty$ to simplify our notation for suprema and infima.

Ex: $(a, +\infty) = \{x \in \mathbb{R} : a < x\} = \{x \in \mathbb{R} : a < x < +\infty\}$

**Def. (Unbounded above/below):** Suppose $S \subseteq \mathbb{R}$ is nonempty.
- If $S$ is not bounded above, write $\sup(S) = +\infty$.
- If $S$ is not bounded below, write $\inf(S) = -\infty$.

**Remark:** Given a nonempty $S \subseteq \mathbb{R}$,

- By defn of supremum and IR
  
  $S$ has a supremum $\iff$ $S$ is bounded above
  $\iff \sup(S) \in \mathbb{R}$

- Similarly,
  
  $S$ doesn't have a supremum $\iff$ $S$ is not bounded above
  $\iff \sup(S) = +\infty$

Using this notation, even though not every set has a supremum, for any nonempty $S \subseteq \mathbb{R}$, $\sup(S)$ has meaning.