Ch 2: Sequences

Recall: functions

**Definition:** A **sequence** is a function whose domain is a set of the form \( \mathbb{Z} \), \( \mathbb{N} \), \( \mathbb{Z}_m \), or \( \mathbb{Z}^+ \). We will study sequences whose range is \( \mathbb{R} \).

Typically, the domain of a sequence will be either \( \mathbb{N} \) or \( \mathbb{Z} \).

**Remark:**

To emphasize that a sequence is a special type of function...

Instead of writing \( f(n) \), we write \( s_n \).

We'll often specify a sequence by listing its values in order, \( (s_1, s_2, s_3, \ldots) \).
Ex: If $s_n = \frac{1}{n}$ for $n \geq 1$, the sequence is $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$

* If $s_n = (-1)^n$ for $n \geq 0$, the sequence is $(1, -1, 1, -1, \ldots)$

Heuristically, a sequence "converges" to some limit $s \in \mathbb{R}$ if the values of $s_n$ stay close to $s$ for large $n$.

Ex: We expect $s_n = \frac{1}{n}$ converges to 0.

We expect $s_n = (-1)^n$ doesn't converge.
Def (convergence):
- A sequence \( s_n \) of real numbers converges to some \( s \in \mathbb{R} \) provided that for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) so that \( n > N \) ensures \( |s_n - s| < \varepsilon \).

- The number \( s \) is the limit of \( s_n \), and we write \( \lim_{n \to \infty} s_n = s \) or \( s_n \to s \).

- A sequence that does not converge to any \( s \in \mathbb{R} \) it is said to diverge.

Remark:
- Recall: \( |b| < a \iff -a < b < a \)
- Thus \( |s_n - s| < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon \)
- \( N \) can depend on \( \varepsilon \).

\[
\begin{align*}
\text{Sn} & \quad 0 \quad a \\
S + \varepsilon_1 & \quad \bullet \\
S + \varepsilon_2 & \quad \bullet \\
S & \quad \bullet \\
S + \varepsilon_2 & \quad \bullet \\
S - \varepsilon_1 & \quad \bullet
\end{align*}
\]
Ex: Consider the sequence \( s_n = \frac{1}{n^2} \). We expect that \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \). Let's prove this!

**Scratchwork:**
\[
\left| \frac{1}{n^2} - 0 \right| < \epsilon \iff \frac{1}{n^2} < \epsilon \iff \frac{1}{\sqrt{\epsilon}} < n
\]

**Proof:** Fix arbitrary \( \epsilon > 0 \). Let \( N = \frac{1}{\sqrt{\epsilon}} \). Then for \( n > N \) we have
\[
\frac{1}{n^2} < \epsilon \iff \frac{1}{\sqrt{\epsilon}} < n \iff \frac{1}{n^2} < \epsilon
\]
Thus \( \lim_{n \to \infty} \frac{1}{n^2} = 0 \). \( \square \)

**Remark:** We could have picked \( N \) to be any number \( \geq \frac{1}{\sqrt{\epsilon}} \), e.g. \( N = \frac{2}{\epsilon^2}, \frac{1}{\sqrt{\epsilon}} + 1, \ldots \)

Ex: Consider the sequence \( s_n = (-1)^n \). We expect that this sequence does not converge. Let's prove it.

**Proof:** Assume, for the sake of contradiction, that \( (-1)^n \) converges to \( s \in \mathbb{R} \). By defn of convergence, for all \( \epsilon > 0 \), there exists \( N \) so that \( n > N \), \( \left| (-1)^n - s \right| < \epsilon \).
Let $\varepsilon = 1$ and choose $N$ so that $n > N$ ensures $|(-1)^n - s| < 1 \iff s - 1 < (-1)^n < s + 1$.

For $n$ even, this implies $1 < s + 1 \implies 0 < s$. For $n$ odd, this implies $s - 1 < -1 \implies s < 0$. This is a contradiction. Thus, $(-1)^n$ diverges. □

Ex: Consider the sequence $s_n = \frac{2n-1}{3n+2}$. What is the limit?

**Scratchwork:**

$$s_n = \frac{2n-1}{3n+2} = \frac{2 - \frac{1}{n}}{3 + \frac{2}{n}}$$

"These get very small as $n \to \infty$"

$$|s_n - \frac{2}{3}| < \varepsilon \iff \left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \varepsilon \iff \left|\frac{6n-3-6n-4}{3(3n+2)}\right| < \varepsilon \iff \frac{7}{3(3n+2)} < \varepsilon$$

$$\iff \frac{7}{4n} < \varepsilon \iff \frac{1}{n} < \varepsilon \iff \frac{1}{\varepsilon} < n$$
**Proof:**

Fix $\varepsilon > 0$ arbitrary and let $N = \frac{1}{\varepsilon}$. Then, if $n > N$, we have

$$\frac{1}{\varepsilon} < n \Rightarrow \frac{7}{3(3n+2)} < \varepsilon \iff \left|6n-3-6n-4\right| < \varepsilon \iff \left|S_n - \frac{2}{3}\right| < \varepsilon.$$

Therefore, $\lim_{n \to \infty} S_n = \frac{2}{3}$.  \[ \square \]

A special type of sequence is a...
**Def (bounded sequence):** A sequence $s_n$ is bounded if there exists $M \in \mathbb{R}$ s.t. $|s_n| \leq M$ for all $n$. 

$$
\iff -M \leq s_n \leq M
$$

**Remark:** A sequence is bounded iff the set 

$$S = \{s_n : n \in \mathbb{N}\}$$

is bounded. (HW3) 

$s_n = (-1)^n$, $S = \{-1, 1\}$, $s_n = \frac{1}{n^2}$, $S = \{1, \frac{1}{4}, \frac{1}{9}, \ldots\}$

**Thm:** Convergent sequences are bounded.

**Pf:** Suppose $s_n$ is a convergent sequence with limit $s$. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $n > N$ ensures $|s_n - s| < \varepsilon$. 

WLOG, we may assume $N > 0$ 

$N \in \mathbb{N}$ so that $n > N$ ensures $|s_n - s| < 1$ 

$$
\iff -1 < s_n - s < 1 \iff s - 1 < s_n < s + 1. \text{ Let } \bigtriangleup \text{ Reverse } \Delta \text{ineq.}
$$
\[ M_0 = \max \{ |s+1|, |s-1| \}. \text{ Then } |s_{n+1} - s_n| \leq |s_{n-1} - s_n| \leq |s_n - s| < 1 \]
\[ \Rightarrow |s_n| < 1 + |s| \]
\[ -M_0 \leq -|s-1| \leq s-1 < s_n < s+1 \leq |s+1| \leq M_0 \]

Thus, if \( n > N \), we have \( |s_n| \leq M_0 \).

\[ \max \{ |s_n| : 1 \leq n \leq N, n \in \mathbb{N}^+ \} \]

Let \( M_1 = \max \{ |s_1|, |s_2|, |s_3|, \ldots, |s_N| \}. \]

Then, for \( 1 \leq n \leq N \), \( |s_n| \leq M_1. \)

Finally, let \( M = \max \{ M_0, M_1 \}. \) Then \( |s_n| \leq M \) \( \forall n \in \mathbb{N} \), so our sequence is bounded.

Remark: The converse is not true, since not all bounded sequences are convergent, e.g., \((-1)^n\).