Lecture 6
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Now, we will prove several limit theorems that will help us find the limits of more complicated sequences by breaking them into parts.

**Thm (limit of sum is sum of limits):** If $s_n$ and $t_n$ are convergent sequences, then \( \lim_{n \to \infty} (s_n + t_n) = \lim_{n \to \infty} s_n + \lim_{n \to \infty} t_n \).

**Ex:** \( \lim_{n \to \infty} \left( \frac{\pi}{n} + \frac{\sqrt{2}}{n^2} \right) = \lim_{n \to \infty} \frac{\pi}{n} + \lim_{n \to \infty} \frac{\sqrt{2}}{n^2} = 0 + 0 = 0 \)

Recall: triangle inequality \( |a+b| \leq |a| + |b| \).

**PG:** Let \( s = \lim_{n \to \infty} s_n \) and \( t = \lim_{n \to \infty} t_n \). Fix \( \epsilon > 0 \).
We must show there exists \( N \in \mathbb{N} \) so that \( n > N \) ensures \( |(s_n + t_n) - (s + t)| < \epsilon \).

Note that \( |(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| \).
Since \( s_n \to s \) and \( t_n \to t \) given \( \varepsilon = \frac{\delta}{2} > 0 \), there exists \( N_s \) and \( N_t \in \mathbb{R} \) so that \( n > N_s \) ensures \( |s_n - s| < \varepsilon \) and \( n > N_t \) ensures \( |t_n - t| < \varepsilon \).

Let \( N = \max \{ N_s, N_t \} \). Then for all \( n > N \),
\[
|s_n + t_n - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon + \varepsilon = \varepsilon.
\]

\( \square \)

Remark: The requirement that \( s_n \) and \( t_n \) are convergent sequences is necessary. For example, \( s_n = (-1)^n \), \( t_n = (-1)^{n+1} \).

Then \( \lim_{n \to \infty} s_n + t_n = 0 \), but \( \lim_{n \to \infty} s_n \) and \( \lim_{n \to \infty} t_n \) do not exist.
**Thm (limit of product is product of limits):** If \( s_n \) and \( t_n \) are convergent sequences, \( \lim_{n \to \infty} s_n t_n = (\lim_{n \to \infty} s_n)(\lim_{n \to \infty} t_n) \)

**Exercise**

Give an example to show that the assumption that \( s_n \) and \( t_n \) are convergent sequences is necessary for the previous theorem to be true.

**Proof:** Let \( s = \lim_{n \to \infty} s_n, \ t = \lim_{n \to \infty} t_n \). Fix \( \varepsilon > 0 \).

We must show there exists \( N \in \mathbb{N} \) so that \( n > N \) ensures \( |s_n t_n - st| < \varepsilon \).

Note that

\[
|s_n t_n - st| = |s_n t_n - s t_n + s t_n - s t| \\
\leq |s_n t_n - s t_n| + |s t_n - s t| \\
= |s_n||t_n - t| + |t||s_n - s|
\]

Since \( s_n \) is a convergent sequence, it is a bounded sequence, that is there exists \( M_s \) so that \( |s_n| \leq M_s \) for all \( n \). Define \( M = \max \{ M_s, |t|, 1 \} \) ensures \( M \geq M_s, M \geq |t|, M > 0 \).
Combining with estimates above, \(|s_n - s| < M|t_n - t| + M|s_n - s|\).
For \(\varepsilon = \frac{\varepsilon}{2M} > 0\), there exists \(N_s\) and \(N_t\) so that \(n > N_s\) ensures \(|s_n - s| < \varepsilon\) and \(n > N_t\) ensures \(|t_n - t| < \varepsilon\). Let \(N = \max\{N_s, N_t\}\). Then for all \(n > N\), \(|s_n - s| < M\varepsilon + M\varepsilon = \varepsilon\). □

**Theorem (limit of quotient is quotient of limits):** If \(s_n\) and \(t_n\) are convergent sequences, \(s_n \neq 0\) for all \(n\), and \(\lim\limits_{n \to \infty} s_n = 0\), then
\[
\lim\limits_{n \to \infty} \left( \frac{t_n}{s_n} \right) = \frac{\lim\limits_{n \to \infty} t_n}{\lim\limits_{n \to \infty} s_n}.
\]

**Proof:** See textbook.
Thm (basic examples):
(a) \( \lim_{n \to \infty} \left( \frac{1}{n} \right)^p = 0 \) if \( p > 0 \)
(b) \( \lim_{n \to \infty} a^n = 0 \) if \( |a| < 1 \)
(c) \( \lim_{n \to \infty} n \cdot n = 1 \)
(d) \( \lim_{n \to \infty} a\cdot n = 1 \) if \( a > 0 \)

Pf: See textbook.