First, we show $\lim_{n \to \infty} S_n$ is an upper bound for $A$. Fix $a \in A$. Then $\exists N_a$ s.t. $n > N_a$ ensures $S_n \geq a$. Thus, for $N > N_b$, $b \in \inf \{S_n : n > N_b\} \leq a$. By the contrapositive of HW4, Q6(a), this gives $\lim_{n \to \infty} S_n = \lim_{n \to \infty} \inf \{S_n : n > N_b\} \geq a$.

Our argument above shows that $\lim_{n \to \infty} S_n = \sup(A)$. Suppose, for the sake of contradiction, that $\sup(A) < \lim_{n \to \infty} S_n$. Then $\exists r \in \mathbb{R}$ s.t. $\sup(A) < r < \lim_{n \to \infty} S_n$. Since $r \in A$, $\exists n \in \mathbb{N} : S_n > r$. Since $\inf \{S_n : n > N_b\}$ is infinite. Thus, $\inf \{S_n : n > N_b\} \leq r$ and $\forall n \in \mathbb{N}$.

This implies $\lim_{n \to \infty} S_n \leq r$, which is a contradiction.

b) $\sup \emptyset = -\infty$
We proceed by induction. For the base case, note that $s_1 = 1$, $s_2 = 2$, $s_3 = \frac{3}{2}$, $s_4 = \frac{5}{3}$. Suppose $s_{2k} \geq s_{2k+1}$ and $s_{2k-1} \leq s_{2k+1}$. Then $s_{2(k+2)} = 1 + \frac{1}{s_{2k+1}} \leq 1 + \frac{1}{s_{2k-1}} = s_{2k}$, so $s_{2k+3} = 1 + \frac{1}{s_{2(k+2)}} \geq 1 + \frac{1}{s_{2k}} = s_{2k+1}$.

Since $s_{2n}$ is decreasing and $s_2 = 2$, we have $s_{2n} \leq 2 \forall n$. Since $s_{2n-1}$ is increasing and $s_1 = 1$, we have $s_{2n-1} \geq 1 \forall n$. Furthermore,

- $s_{2n} = 1 + \frac{1}{s_{2n-1}} \geq 1$ and $s_{2n+1} = 1 + \frac{1}{s_{2n}} \leq 2$. This shows

Since $s_n \leq 2$ for all $n$, the subsequence of even terms and the subsequence of odd terms are both bounded and monotone. Hence, they both converge. Let $\lim_{k \to \infty} s_{2k} = S_{\text{even}}$ and $\lim_{k \to \infty} s_{2k-1} = S_{\text{odd}}$. 
Note that:
\[ S_{2k+1} = 1 + \frac{1}{S_{2k}} = 1 + \frac{1}{1 + \frac{1}{S_{2k-1}}}. \]
Since \( S_{2k-1} \geq 1 \), \( S_{2k+1} = 1 \). Thus, applying the limit theorems (quotient, sum), we have \( S_{2k+1} = \lim_{k \to \infty} S_{2k+1} = 1 + \frac{1}{S_{2k+1}} \).

Thus, soda solved \( (S_{2k+1} - 1) = (1 + \frac{1}{S_{2k+1}})^{-1} \)
\( \leq (S_{2k+1} - 1)(1 + \frac{1}{S_{2k+1}}) = 1 \leq S_{2k+1} + 1 - 1 + \frac{1}{S_{2k+1}} = 1 \)
\( \Rightarrow S_{2k+1} - S_{2k} - 1 = 0. \) By the quadratic formula and the fact that \( S_{2k+1} \in [1, 2] \), we obtain \( S_{2k+1} = 9. \)

Finally,
\[ S_{2k+1} = 1 + \frac{1}{S_{2k-1}}. \]
Again, applying the limit theorems, we obtain \( S_{2k} = 1 + \frac{1}{S_{2k-1}} \Rightarrow S_{2k} = 9. \)

\( \text{Fix } \varepsilon > 0. \text{ Choose } N_{\text{even}} \text{ so } k > N_{\text{even}} \text{ ensures } |s_{2k} - 9| < \varepsilon \) and \( N_{\text{odd}} \) so \( k > N_{\text{odd}} \text{ ensures } |s_{2k-1} - 9| < \varepsilon \).
Let \( N = 2 \cdot \max\{N_{\text{even}}, N_{\text{odd}}\} \). Then
Let $n > N$ ensure that either $n = 2k$ and $k > N_{\text{even}}$ or $n = 2k-1$ and $k > N_{\text{odd}}$. In either case $|s_n - \varphi| < \varepsilon$.

3) 

a) $\limsup_{n \to +\infty} s_n = \lim_{N \to +\infty} \sup \{s_n : n > N\}$

b) Suppose $\limsup_{n \to +\infty} |s_n| = 0$. Since $|s_n| \geq 0$, $\liminf_{n \to +\infty} |s_n| = 0$. Thus $\lim_{n \to +\infty} |s_n| = 0$ and $\lim_{n \to +\infty} -|s_n| = 0$. Since $-|s_n| \leq s_n \leq |s_n|$, the squeeze lemma ensures $\lim_{n \to +\infty} s_n = 0$.

c) Suppose $\lim_{n \to +\infty} s_n = 0$. As shown on Midterm 1, $\lim_{n \to +\infty} |s_n| = |\lim_{n \to +\infty} s_n| = |0| = 0$. Thus $\limsup_{n \to +\infty} |s_n| = 0$.

4) 

1. 
   (a) 
   (b) 

2. 
   (i) True 
   (ii) False - consider $s_n = (-1)^n$
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1. \( \limsup_{n \to \infty} a_n = 1 \), \( \liminf_{n \to \infty} a_n = -1 \)
   a. True
   b. False

2. a. No
   b. No