Homework 1 Solutions
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① ⓐ Base case: When $n=1$, we have $3 = 4\cdot 1^2 - 1$.

Inductive step: Suppose $3+11+\dots+(8n-5) = 4n^2 - n$.

Then $3+11+\dots+(8n-5)+(8(n+1)-5) = (4n^2-n)+(8(n+1)-5)$

$= 4n^2 - n + 8n + 3 = 4(n^2+2n+1) - (n+1) = 4(n+1)^2 - (n+1)$,

which completes the proof.

ⓑ Base case: When $n=1$, we have $1+\frac{1}{2} = 2 - \frac{1}{2}$.

Inductive step: Suppose $1+\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n} = 2-\frac{1}{2^n}$.

Then $1+\frac{1}{2}+\dots+\frac{1}{2^n}+\frac{1}{2^{n+1}} = 2-\frac{1}{2^n}+\frac{1}{2^{n+1}}$

$= 2+\left(\frac{1}{2^{n+1}}-\frac{2}{2^{n+1}}\right) = 2-\frac{1}{2^{n+1}}$, which completes the proof.

② ⓐ Base case: When $n=2$, we have $2^2 = 4 > 3 = 2+1$.

Inductive step: Suppose $n^2 > n+1$. Then

$(n+1)^2 = n^2+2n+1 > (n+1)+2n+1 \geq (n+1)+1$,

which completes the proof.

ⓑ Base case: When $n=4$, we have $4! = 24 > 16 = 4^2$.

Inductive step: Suppose $n! > n^2$. Then

$(n+1)! = (n+1)\cdot n! > (n+1)n^2$. By part (a), $n^2 > n+1$ for all $n \geq 2$. Therefore, we may continue the previous inequality to obtain $(n+1)n^2 > (n+1)^2$. Hence $(n+1)! > (n+1)^2$, which completes the proof.

③ Assume for the sake of contradiction that $q > p$. By the proposition from class, there exists $r \in \mathbb{Q}$ so that $q > r > p$. By assumption, since $r > p$, we must have $q \leq r$. This contradicts the fact that $q > r$. Therefore, we must have $q \leq p$.

(4)

(i) Case 1: $a \geq 0$

Then $|a| = a \geq 0$

Case 2: $a \leq 0$

Then $-a \geq 0$, so $|a| = -a \geq 0$.

(ii) Case 1: $ab \geq 0$

Then either $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$. In the first case,

$$|ab| = ab = |a||b|.$$

In the second case,

$$|ab| = ab = (-a)(-b) = |a||b|.$$

Case 2: $ab \leq 0$

Then either $a \geq 0$ and $b \leq 0$ or $a \leq 0$ and $b \geq 0$. By commutativity of multiplication, we may assume WLOG that $a \geq 0$ and $b \leq 0$. Thus,

$$|ab| = -ab = a(-b) = |a||b|.$$

(iii) Case 1: $a \geq 0$

Then $|a| = a$, so $|a| \geq a$. Furthermore, since $-a \leq 0$, part (i) ensures $|a| \geq 0 \geq -a$.

Case 2: $a \leq 0$

Since $-a \geq 0$ and part (ii) ensures $|-a| = |-1||a| = |a|$, by Case 1, we have

$$|a| = |-a| \geq -a \quad \text{and} \quad |a| = |-a| \geq -(-a) = a.$$

(iv) Case 1: $a+b \geq 0$

Then, by part (iii),

$$|a+b| = a+b \leq |a| + |b|.$$

Case 2: $a+b \leq 0$

Then, by part (iii),

$$|a+b| = -(a+b) = -a-b \leq |a| + |b|.$$

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a) Suppose $|b| \leq a$.
By 4 iii, $b \leq |b| \leq a$ and $-b \leq |b| \leq a$.
Multiplying the second inequality on both sides by -1, we obtain $-a \leq b$. Thus $-a \leq b \leq a$.

Now, suppose $-a \leq b \leq a$. Then $-a \leq b \Rightarrow -b \leq a$. Since $b \leq a$ and $-b \leq a$, the definition of the absolute value ensures $|b| \leq a$.

b) First, we will show $|a| - |b| \leq |a-b|$ (*) for all $a, b \in \mathbb{R}$. This is a consequence of the triangle inequality (Q4 iv), since

$$|a| = |(a-b)+b| \leq |a-b| + |b|.$$

Now, note that since a and b were arbitrary, we also have

$$|b| - |a| \le |a-b| \quad (**)$$

for all $a, b \in \mathbb{R}$. Combining $(*)$ and $(**)$, by the definition of the absolute value, we obtain

$$||a| - |b|| \le |a-b|.$$

6 a Applying the result from Q5 a,

$$|a-b| \le c \overset{Q5 a}{\iff} -c \le a-b \le c \iff b-c \le a \le b+c.$$

b Note that, if either $|a-b| < c$ or if $b-c < a < b+c$, we must have $c > 0$.

Thus, we assume $c > 0$.
It suffices to show that
$|a-b| \ne c \iff a \ne b-c$ and $a \ne b+c$.

Case 1: $a \ge b$.
Since $c \ge 0$, we always have $a \ne b-c$.

Furthermore,
$$|a-b| \ne c \iff a-b \ne c \iff a \ne b+c$$

Case 2: $b \geq a$.

Since $c \geq 0$, we always have $a \neq b+c$.

Furthermore,

$|a-b| \neq c \iff b-a \neq c \iff a \neq b-c$.