Math 117: Homework 1

Due Friday, January 12th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

(a) Prove by induction that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$.

(b) Prove by induction that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

Question 2*

We now describe a useful extension of the principle of mathematical induction. Fix $m \in \mathbb{N}$. Given a list of propositions $P_m, P_{m+1}, P_{m+2}, \ldots$, if you can show that

(i) P_m is true;

(ii) for all $n \in \mathbb{N}$ with $n \geq m$, if P_n is true then P_{n+1} is true;

then $P_m, P_{m+1}, P_{m+2} \ldots$ are all true.

Use this extension of the principle of mathematical induction to prove the following statements:

(a) $n^2 > n + 1$ for all natural numbers $n \geq 2$.

(b) $n! > n^2$ for all natural numbers $n \geq 4$. (Recall that $n! = n(n-1) \cdot \cdots \cdot 2 \cdot 1$. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.)

Question 3*

Suppose $p, q \in \mathbb{Q}$. Suppose that for all $s \in \mathbb{Q}$ with $s > p$, we have $q \leq s$. Prove that $q \leq p$.
(Hint: Prove the result by contradiction, using the fact that, between any two rational numbers, there is a rational number.)

Question 4*

Use the definition of the absolute value to prove the following results for all $a, b \in \mathbb{R}$. You may use all facts you learned about $\mathbb{R}$ in previous courses without further justification.

(i) $|a| \geq 0$

(ii) $|ab| = |a|\ |b|$

(iii) $|a| \geq a$ and $|a| \geq -a$

(iv) $|a + b| \leq |a| + |b|$

Question 5

Use the definition of the absolute value to prove the following results for all $a, b \in \mathbb{R}$. You may use all facts you learned about $\mathbb{R}$ in previous courses without further justification.

(a) For any $a, b \in \mathbb{R}$, prove that $|b| \leq a$ if and only if $-a \leq b \leq a$.

(b) For any $a, b \in \mathbb{R}$, prove that $||a| - |b|| \leq |a - b|$. This is known as the *reverse triangle inequality*.

Question 6*

Use the definition of the absolute value to prove the following results for any $a, b, c \in \mathbb{R}$. You may use all facts you learned about $\mathbb{R}$ in previous courses without further justification.
We will use these inequalities repeatedly throughout the course.

(a) Prove that $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$

(b) Prove that $|a - b| < c$ if and only if $b - c < a < b + c$.

For each part, draw a picture of the real number line to illustrate the statement that you proved.