Homework 2 Solutions

① (a) By (M3), $1^2=1$, so by part (iv) of the theorem, $0 \leq 1^2 = 1$. By (M3), $0 \neq 1$, so $0<1$.

(b) By part (v) of the Theorem, $0<\frac{1}{b}$ and $0<\frac{1}{a}$, so it suffices to show $\frac{1}{b}<\frac{1}{a}$.

Note that $a<b \Rightarrow a \leq b$

$$\overset{(O5)}{\Rightarrow} a\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right) \leq b\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right) \overset{(M2)}{\Rightarrow} a\cdot\left(\frac{1}{a}\cdot\frac{1}{b}\right) \leq b\left(\frac{1}{b}\cdot\frac{1}{a}\right)$$

$$\overset{(M1)}{\Rightarrow} \left(a\cdot\frac{1}{a}\right)\cdot\frac{1}{b} \leq \left(b\cdot\frac{1}{b}\right)\cdot\frac{1}{a} \overset{(M4)}{\Rightarrow} 1\cdot\frac{1}{b} \leq 1\cdot\frac{1}{a}$$

$$\overset{(M2)}{\Rightarrow} \frac{1}{b}\cdot 1 \leq \frac{1}{a}\cdot 1 \overset{(M3)}{\Rightarrow} \frac{1}{b} \leq \frac{1}{a}.$$

It remains to show $\frac{1}{b}\neq\frac{1}{a}$. Suppose for the sake of contradiction that $\frac{1}{b}=\frac{1}{a}$.

Then, $\frac{1}{b}=\frac{1}{a} \Rightarrow \frac{1}{b}(ab)=\frac{1}{a}(a\cdot b)$

$$\overset{(M1 \text{ and } M2)}{\Rightarrow} \left(\frac{1}{b}\cdot b\right)\cdot a = \left(\frac{1}{a}\cdot a\right)\cdot b \overset{(M4)}{\Rightarrow} 1\cdot a = 1\cdot b \overset{(M2, M3)}{\Rightarrow} a=b,$$

which contradicts that $a<b$. Therefore $\frac{1}{b}\neq\frac{1}{a}$

② If $s_0=\max(S)$, then $s_0 \in S$ and $s_0 \geq s$ $\forall s\in S$. Consequently, s_0 is an upper bound of S. Suppose s_1 is another upper bound of S. Since $s_0\in S$, $s_1 \geq s_0$. Thus, s_0 is the least upper bound of S, so $s_0=\sup(S)$.

③ mild notational difference: replace a_i with y_i.

Base case: When $n=1$, $|a_1| \leq |a_1|$.

Inductive step: Assume $|a_1+a_2+\ldots+a_n| \leq |a_1|+\ldots+|a_n|$

By the triangle inequality (as stated in part ⓐ) with $x=a_1+a_2+\ldots+a_n$ and $y=a_{n+1}$

$|a_1+a_2+\ldots+a_n+a_{n+1}| \leq |a_1+a_2+\ldots+a_n| + |a_{n+1}|$.

By the inductive hypothesis, the right hand side is bounded above by $|a_1|+|a_2|+\ldots+|a_n|+|a_{n+1}|$, which completes the proof.

mild notational difference: replace S with T

④ ⓐ By definition, $\sup(S) \geq s$ for all $s \in S$.
Thus, if $\sup(S) \in S$, it is the largest element of S and $\max(S) = \sup(S)$.

ⓑ Suppose S has a maximum M_0.
By Q2, $M_0 = \sup(S)$. This contradicts the fact that $\sup(S) \notin S$. Therefore, S must not have a maximum.

5) a) Let s be an element of S. Since $\inf(S)$ is a lower bound for S, $\inf(S) \le s$. Since $\sup(S)$ is an upper bound for S, $s \le \sup(S)$. Therefore, $\inf(S) \le \sup(S)$.

b) We will show $S = \{\inf(S)\}$, so there is one element in the set. Since $\inf(S) = \sup(S)$, $\inf(S)$ is both an upper and lower bound for S. In particular, for any $s \in S$, $\inf(S) \le s$ and $\inf(S) \ge s$. Thus $\inf(S) = s$ for all $s \in S$. This shows $S = \{\inf(S)\}$.

mild notational change: A, B becomes S, T

7 (a) Since $\inf(T)$ is a lower bound for the set T, if $S \subseteq T$, then $\inf(T)$ is also a lower bound for the set S. Since $\inf(S)$ is the greatest lower bound of S, $\inf(T) \leq \inf(S)$. The fact that $\sup(S) \leq \sup(T)$ follows from an analogous argument. The fact that $\inf(S) \leq \sup(S)$ follows from Q5 (a).

(b) Since $\sup(S)$ is an upper bound for S and $\sup(T)$ is an upper bound for T, $\max\{\sup(S), \sup(T)\}$ is an upper bound for $S \cup T$. Thus, since $\sup(S \cup T)$ is the least upper bound for $S \cup T$, $\sup(S \cup T) \leq \max\{\sup(S), \sup(T)\}$.

By Q7 (a), since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $\sup(S) \leq \sup(S \cup T)$ and $\sup(T) \leq \sup(S \cup T)$. Thus $\max\{\sup(S), \sup(T)\} \leq \sup(S \cup T)$.

Combining these two inequalities, we conclude $\max\{\sup(S), \sup(T)\} = \sup(S \cup T)$.

(7) (a) Since S is bounded below, $\exists\, m_0 \in \mathbb{R}$ s.t. $s \geq m_0 \ \forall s \in S$. This implies $-m_0 \geq -s \ \forall s \in S$, so $-S$ is bounded above.

(b) Since S is nonempty, so is $-S$. Since $-S$ is bounded above, by definition of the real numbers, it has a supremum, $\sup(-S)$.

(c) Since $\sup(-S)$ is an upper bound for $-S$, $-s \leq \sup(-S)$ for all $s \in S$, hence $s \geq -\sup(-S)$ for all $s \in S$. Therefore $-\sup(-S)$ is a lower bound for S, and it suffices to show it is the greatest lower bound.

Suppose for the sake of contradiction that m_0 is a lower bound for S with $m_0 > -\sup(-S)$. As argued in part (a), $-m_0$ is an upper bound for $-S$. Furthermore, $m_0 > -\sup(-S)$ implies $-m_0 < \sup(-S)$. This is a contradiction, since $\sup(-S)$ is the least upper bound for $-S$.

Therefore $-\sup(-S)$ must be the greatest lower bound of $-S$.

notational change: $A+B$ becomes $S+T$

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Step 1: Show that for all $t \in T$, $\inf(S+T)-t$ is a lower bound for S.

By defn of $S+T$ and the infimum, $\inf(S+T)$ is a lower bound for $S+T$, so $s+t \geq \inf(S+T) \iff s \geq \inf(S+T)-t$ for all $s \in S$, $t \in T$. Thus, for all $t \in T$, $\inf(S+T)-t$ is a lower bound for S.

Step 2: Show that $\inf(S+T)-\inf(S)$ is a lower bound for T.

By Step 1, for all $t \in T$, $\inf(S+T)-t$ is a lower bound for S. By defn, $\inf(S)$ is the greatest lower bound of S. Thus, $\inf(S) \geq \inf(S+T)-t$

$\iff t \geq \inf(S+T)-\inf(S)$ for all $t \in T$.

Since $\inf(S+T)-\inf(S)$ is a lower bound for T and $\inf(T)$ is the greatest lower bound,

$$\inf(T) \geq \inf(S+T)-\inf(S).$$
$$\Updownarrow$$
$$\inf(S)+\inf(T) \geq \inf(S+T). \quad (*)$$

It remains to prove the opposite inequality. Since $\inf(S)$ and $\inf(T)$ are lower bounds for S and T, for all $s \in S$ and $t \in T$, $\inf(S) \leq s$ and $\inf(T) \leq t \Rightarrow \inf(S)+\inf(T) \leq s+t$. Thus, $\inf(S)+\inf(T)$ is a lower bound for $S+T$. Since $\inf(S+T)$ is the greatest lower bound,

$$\inf(S)+\inf(T) \leq \inf(S+T). \quad (**)$$

Thus, combining inequalities $(*)$ and $(**)$, we obtain

$$\inf(S)+\inf(T)=\inf(S+T). \quad \square$$

slight notational change: $A=S$, $B=T$

(9) (a) Since $s \le t$ for all $s \in S$ and $t \in T$, any $t \in T$ is an upper bound for S and any $s \in S$ is a lower bound for T. Hence, S is bounded above and T is bounded below.

(b) As shown in part (a), any $t \in T$ is an upper bound for S. Since $\sup(S)$ is the least upper bound, $\sup(S) \le t$ for all $t \in T$. Thus, $\sup(S)$ is a lower bound for T, and since $\inf(T)$ is the greatest lower bound, $\sup(S) \le \inf(T)$.

(c) $S=[0,1]$, $T=[1,2]$

(d) $S=[0,1)$, $T=(1,2]$

(10) Throughout, we use S to denote the set under consideration.

(a) $\sup(S)=\sqrt{2}$, $\inf(S)=-\sqrt{2}$

(b) $\sup(S)=\pi$, $\inf(S)=-1$

(c) $\sup(S)=\inf(S)=1$

(d) S is not bounded above, $\inf(S)=1$

(e) $\sup(S)=1$, $\inf(S)=0$

(f) $\sup(S)=1$; $\inf(S)=-1$

(g) $S=[-1,1]$, so $\sup(S)=1$ and $\inf(S)=-1$

(11) (a) $\sup(S)=1$, $\inf(S)=0$

(b) S is not bounded above, $\inf(S)=0$

(c) S is not bounded above, $\inf(S)=0$

(d) S is neither bounded above or below

(e) $S=\{0\}$, so $\sup(S)=\inf(S)=0$

(f) S is not bounded above, $\inf(S)=2^{1/3}$

(g) $\sup(S)=\inf(S)=0$