MATH 117: HOMEWORK 2

Due Friday, January 19th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

The textbook proves the following result (see Theorem 3.2, p16):

THEOREM 1. *Suppose F is an ordered field. Then the following properties hold for all $a, b, c \in F$:*

(i) *if $a \leq b$, then $-b \leq -a$;*
(ii) *if $a \leq b$ and $c \leq 0$, then $bc \leq ac$;*
(iii) *if $0 \leq a$ and $0 \leq b$, then $0 \leq ab$;*
(iv) *$0 \leq a^2$, where a^2 is an abbreviation for $a \cdot a$;*
(v) *if $0 < a$, then $0 < 1/a$.*

Using this theorem and the definition of an ordered field, prove the following results for an ordered field F. Explain which properties of a field you use (A1-A4, M1-M4, DL), which properties of an ordered field you use (O1-O5), and which parts of Theorem 1 you use (i-v).

(a) For 0 and 1 as in the definition of a field, $0 < 1$.

(b) For all $a, b \in F$, if $0 < a < b$, then $0 < 1/b < 1/a$.

Question 2*

Prove that if S is a nonempty subset of $\mathbb{R}$ with maximum s_0, then $\sup(S) = s_0$.

Question 3

Prove by induction that, for $n \in \mathbb{N}$ and $y_1, y_2, \ldots, y_n \in \mathbb{R}$,

$$|y_1 + y_2 + \cdots + y_n| \leq |y_1| + |y_2| + \cdots + |y_n|.$$

Question 4*

Let T be a nonempty subset of $\mathbb{R}$ that is bounded above.

(a) Prove that $\sup T \in T$ implies $\sup T = \max T$.

(b) Prove that if $\sup T$ is not an element of T, then the maximum of T does not exist.

Question 5

Let S be a nonempty bounded subset of $\mathbb{R}$.

(a) Prove that $\inf S \leq \sup S$.

(b) What can you say about the number of elements in S if $\inf S = \sup S$? Justify your answer.

Question 6*

Suppose A and B are nonempty bounded subsets of $\mathbb{R}$.

(a) Prove that if $A \subseteq B$, then $\inf B \leq \inf A \leq \sup A \leq \sup B$.

(b) Prove $\sup(A \cup B) = \max\{\sup A, \sup B\}$. (Note: for this part, do not assume $A \subseteq B$.)

Question 7*

Consider the following proposition:

PROPOSITION 2. *Every nonempty subset S of $\mathbb{R}$ that is bounded below has an infimum.*

(See also Corollary 4.5 on p23 of the textbook.)

This question will lead you through the proof of the proposition.

(a) Suppose S is as in the proposition above. Define the set $-S = \{-s : s \in S\}$. Show that $-S$ is bounded above.

(b) Use the definition of $\mathbb{R}$ to prove that $-S$ has a supremum, $\sup(-S)$.

(c) Prove that $-\sup(-S)$ is the infimum of S.

Question 8

Let A and B be nonempty bounded subsets of $\mathbb{R}$, and define $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Prove $\sup(A + B) = \sup A + \sup B$.

Hint: Show that for all $b \in B$, $\sup(A+B)-b$ is an upper bound for A. Hence $\sup A \leq \sup(A+B)-b$ for all $b \in B$. Then show $\sup(A + B) - \sup A$ is an upper bound for B.

Question 9

Let A and B be nonempty subsets of $\mathbb{R}$ with the following property: $a \leq b$ for all $a \in A$ and $b \in B$.

(a) Show that A is bounded above and B is bounded below.

(b) Prove $\sup A \leq \inf B$.

(c) Given an example of A and B satisfying the above property where $A \cap B$ is nonempty.

(d) Give an example of A and B satisfying the above property where $\sup A = \inf B$ and $A \cap B$ is the empty set. You do not need to justify your example with a proof.

Question 10*

For each of the sets below, answer the following questions: Is it bounded above? If so, what is its supremum? Is it bounded below? If so, what is its infimum? You do not need to justify your answers.

(a) $[-\sqrt{2}, \sqrt{2}]$

(b) $\{-1, 0, e, \pi\}$

(c) $\{1\}$

(d) $\cup_{n=1}^{\infty}[2n-1, 2n)$

(e) $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$

(f) $\{x \in \mathbb{R} : x^2 < 1\}$

(g) $\cap_{n=1}^{\infty}(-1 - \frac{1}{n}, 1 + \frac{1}{n})$

Question 11

Follow the same instructions as in Question 10 for the following sets:

(a) $\left\{\frac{1}{n^2} : n \in \mathbb{N}\right\}$

(b) $\left\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

(c) $\{q \in \mathbb{Q} : q \geq 0\}$

(d) $\{q \in \mathbb{Q} : q^2 \geq 0\}$

(e) $\cap_{n=1}^{\infty}\left(\frac{-1}{n}, \frac{1}{n}\right)$

(f) $\{x \in \mathbb{R} : x^3 \geq 2\}$

(g) $\{\sin(n\pi) : n \in \mathbb{N}\}$