MATH 117: HOMEWORK 3

Due Friday, January 26th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Suppose $S \subseteq \mathbb{R}$ is nonempty and bounded above. Prove that a is the supremum of S if and only if a is an upper bound for S and, for all $\epsilon > 0$, there exists $s \in S$ so that $s > a - \epsilon$.

Question 2

Consider $x, y \in \mathbb{R}$ satisfying $x, y \in [1, 2]$. Suppose $x^2 < 2$ and $y^2 > 2$.

(a) Suppose $0 < \epsilon < 1$. Prove that $(x + \epsilon)^2 \leq x^2 + 5\epsilon$ and $(y - \epsilon)^2 \geq y^2 - 4\epsilon$.

(b) Prove that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $x^2 + 5\epsilon_1 < 2$ and $y^2 - 4\epsilon_2 > 2$.

(c) Use parts (a) and (b) to show that there exists $\epsilon_1, \epsilon_2 \in (0, 1)$ so that $(x + \epsilon_1)^2 < 2$ and $(y - \epsilon_2)^2 > 2$.

Question 3*

Consider an ordered field F. For any $a \in F$, recall that a^2 is an abbreviation for $a \cdot a$.

Fix $a \in F$ with $a \geq 0$. Consider the set $S = \{c \in F : c \geq 0,\ c^2 \leq a\}$

(a) Prove that S is nonempty and bounded above.

(b) Suppose $F = \mathbb{R}$. Explain why, in this case, the supremum of S exists.

Question 4*

Consider the set $S = \{c \in \mathbb{R} : c \geq 0,\ c^2 \leq 2\}$. Let $b = \sup(S)$.

(a) Prove that $b \in [1, 2]$.

(b) Prove that $b^2 \geq 2$. (Hint: proceed by contradiction, using question 2).

(c) Prove that $b^2 \leq 2$. (Hint: proceed by contradiction, using question 2).

Combining parts (b) and (c), we see that $b^2 = 2$.

In this way, we have shown there exists a real number $b > 0$ so that $b^2 = 2$. We can now define the symbol $\sqrt{2}$ by setting $\sqrt{2} := b$. In this way, we have proved $\sqrt{2} \in \mathbb{R}$. Combining this with our result from class that $\sqrt{2} \notin \mathbb{Q}$, we see that $\sqrt{2} \in \mathbb{I}$, where $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ is the set if *irrational numbers*.

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Question 5*

(a) Prove the following, using the definition of convergence:

$$\lim_{n \to +\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1. \\ 1 & \text{if } a = 1. \end{cases}$$

(Hint: recall that the natural logarithm $\log(x)$ is an increasing function for $x > 0$: that is, for any $x, y > 0$, $x \leq y \iff \log(x) \leq \log(y)$.)

(b) If $a \leq -1$ prove that the sequence does not converge.

Question 6

Fix $a \in \mathbb{R}$ and consider the collection of rational numbers $S = \{q \in \mathbb{Q} : a \leq q\}$.

(a) Suppose the underlying field is either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{Q}$. For which values of a does the minimum of S exist? Justify your answer with a proof.

(b) Suppose the underlying field is $\mathbb{F} = \mathbb{R}$. Prove that $\inf(S) = a$.

Question 7

Given $s, t \in \mathbb{R}$, consider the set $(s, t] = \{x \in \mathbb{R} : s < x \leq t\}$. Find the maximum, minimum, supremum, and infimum of the set or state that they do not exist. Justify your answers with proofs.

Question 8

Prove that if $s > 0$, then there exists $n \in \mathbb{N}$ satisfying $\frac{1}{n} < s < n$.

Question 9*

Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$. (Compare to Question 3 on HW 1.)

Question 10*

(a) State the definition of what it means for a sequence s_n to converge to a limit s

(b) State the definition of what it means for a sequence s_n to *not* converge to a limit s

(c) Use the definition of a convergent sequence to prove that $\lim_{n \to +\infty} \frac{n-3}{n^2+9} = 0$.

(d) Use the definition of a convergent sequence to prove that the sequence $s_n = (n+1)^2 - 2$ does not converge.

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Question 11

Determine if the following sequences converge. Justify your answer with a proof.

(a) $a_n = \frac{7n-19}{3n+7}$

(b) $b_n = \sin\left(\frac{n\pi}{3}\right)$

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