Homework 3 Solutions

① Suppose $a = \sup(S)$. By definition, a is an upper bound of S. Fix $\varepsilon > 0$. Then $a - \varepsilon < a$. Thus, $a - \varepsilon$ must not be an upper bound of S. Hence, there exists $s \in S$ s.t. $a - \varepsilon < s$.

Suppose a is an upper bound of S and, for all $\varepsilon > 0$, $\exists s \in S$ s.t. $s > a - \varepsilon$. Let m be an upper bound of S. Assume, for the sake of contradiction, that $m < a$. Then $\varepsilon := a - m > 0$. Thus, $\exists s \in S$ s.t. $s > a - \varepsilon = m$. This contradicts that m was an upper bound. Therefore $m \geq a$. This shows a is the least upper bound or supremum of S.

(2)

(a) We estimate as follows:

$$(x+\varepsilon)^2 = x^2 + 2\varepsilon x + \varepsilon^2 \overset{x\le 2}{\le} x^2 + 4\varepsilon + \varepsilon^2 \overset{\varepsilon\le 1 \Rightarrow \varepsilon^2 \le \varepsilon}{\le} x^2 + 5\varepsilon$$

$$(y-\varepsilon)^2 = y^2 - 2\varepsilon y + \varepsilon^2 \overset{y\le 2}{\ge} y^2 - 4\varepsilon + \varepsilon^2 \overset{\varepsilon^2 \ge 0}{\ge} y^2 - 4\varepsilon.$$

(b) Since $x^2 < 2$, $\tilde{\varepsilon}_1 := 2 - x^2 > 0$. Let $\varepsilon_1 = \frac{\tilde{\varepsilon}_1}{10}$.
Then $x^2 + 5\varepsilon_1 = x^2 + \frac{\tilde{\varepsilon}_1}{2} < x^2 + \tilde{\varepsilon}_1 = 2$.

Since $y^2 > 2$, $\tilde{\varepsilon}_2 = y^2 - 2 > 0$. Let $\varepsilon_2 = \frac{\tilde{\varepsilon}_2}{8}$.
Then $y^2 - 4\varepsilon_2 = y^2 - \frac{\tilde{\varepsilon}_2}{2} > y^2 - \tilde{\varepsilon}_2 = 2$.

(c) We estimate as follows:

$$(x+\varepsilon_1)^2 \overset{(a)}{\le} x^2 + 5\varepsilon_1 \overset{(b)}{<} 2$$

$$(y-\varepsilon_2)^2 \overset{(a)}{\ge} y^2 - 4\varepsilon_2 \overset{(b)}{>} 2.$$

(3)

(a) As shown in class, $0^2 = 0$. Thus $0 \in S$, so $S \neq \emptyset$.

Define $m = \max\{a, 1\}$. Suppose $c \in S$. If $c \leq 1$, then $c \leq 1 \leq m$. If $c \geq 1$, then by (05), $c^2 \geq c$, so $c \leq c^2 \leq a \leq m$. This shows S is bounded above by m. By definition, S is bounded below by 0.

(b) By definition of $\mathbb{R}$, for any nonempty subset of $\mathbb{R}$ that is bounded above, the supremum exists.

(4)

(a) In (3)(a), we showed $m = \max\{2, 1\} = 2$ is an upper bound for S. Thus $b \leq 2$. Since $1 \geq 0$ and $1^2 \leq 2$, $1 \in S$, so $b \geq 1$.

(b) If $b^2 < 2$, by (2)(c) $\exists\ \varepsilon_1 \in (0,1)$ s.t. $(b+\varepsilon_1)^2 < 2$. Thus $b + \varepsilon_1 \in S$. This contradicts the fact that b is an upper bound of S.

(c) If $b^2 > 2$, by (2)(c), $\exists\, \varepsilon_2 \in (0,1)$ s.t. $(b-\varepsilon_2)^2 > 2$. Thus, if $c > b - \varepsilon_2$,

$$c^2 \overset{c \geq 0}{\geq} c(b-\varepsilon_2) \overset{b-\varepsilon_2 \geq 0}{\geq} (b-\varepsilon_2)^2 > 2,$$

so $c \notin S$. By Q1, this contradicts that b is the supremum.

(5) We first show $\lim a^n = 0$ if $|a|<1$.

(a) Note that, if $a=0$, then for all $\varepsilon>0$ and any $N\in\mathbb{R}$, $n>N$ ensures $|a^n-0| = 0<\varepsilon$. Thus, $\lim a^n=0$.

Now, suppose that $a\neq 0$. Fix $\varepsilon>0$. Note that $|a^n-0|<\varepsilon$
$\iff |a^n|<\varepsilon \iff |a|^n<\varepsilon \iff n\log(|a|)<\log\varepsilon$

since $|a^n|=|a|^n$

since $|a|<1$ ensures $\log(|a|)<0$

$\iff n>\dfrac{\log\varepsilon}{\log(|a|)}$. Let $N=\dfrac{\log\varepsilon}{\log(|a|)}$.

Then $n>N$ ensures $|a^n-0|<\varepsilon$, so $\lim a^n=0$.

We now show $\lim a^n = 1$ if $a=1$.
Note that if $a=1$, then $a^n=1$ for all n.
Fix $\varepsilon>0$ and choose $N=1$. Then $n>N$ ensures $|a^n-1| = 0<\varepsilon$.

(b) We conclude by showing a^n does not converge if $a\leq -1$.

Suppose for the sake of contradiction that a^n converges to some $a \in \mathbb{R}$. Let $\varepsilon = 1$. Then there exists N s.t. $n > N$ ensures $|a^n - a| < 1 \iff a-1 < a^n < a+1$.
For n even, $a^n = |a|^n > 1$ so $1 < a+1 \Rightarrow 0 < a$.
For n odd, $a^n = -|a|^n < -1$, so $a-1 < -1 \Rightarrow a < 0$.
This is a contradiction, since no $a \in \mathbb{R}$ can satisfy both $a > 0$ and $a < 0$.

6

(a) We will show that the minimum of S exists if $a \in \mathbb{Q}$ and does not exist if $a \in \mathbb{R} \setminus \mathbb{Q}$.

Suppose $a \in \mathbb{Q}$. By definition of S, $a \in S$. Furthermore, for any $s \in S$, $a \leq S$. Thus $\min(S) = a$.

Suppose $a \in \mathbb{R} \setminus \mathbb{Q}$. Assume, for the sake of contradiction, that the minimum of S exists, and $\min(S) = s_0$. Since $s_0 \in S$, we have $s_0 \geq a$. However, since $a \notin \mathbb{Q}$, we must have $s_0 > a$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $s_0 > r > a$. By definition of S, we must have $r \in S$. This contradicts that s_0 was the minimum of S. Thus, the minimum of S must not exist.

(b) We will show that $\inf(S) = a$. By definition of S, $a \leq s$ for all $s \in S$, so a is a lower bound for S. Suppose $m_0 \in \mathbb{R}$ is another lower bound of S. Assume for the sake of contradiction that $m_0 > a$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ so that $m_0 > r > a$. Thus, $r \in S$, which contradicts the definition of m_0 as a lower bound of S. This shows $m_0 \leq a$, so a is the greatest lower bound.

(7) Let $S=(a,b]$.

- $\max(S)=b$, since by defn, b is the largest element in S
- $\sup(S)=b$. b is an upper bound for S and since $b\in S$, no number smaller than b can be an upper bound. Thus b is the least upper bound.
- The minimum of S does not exist. Suppose, for the sake of contradiction that $\min(S)=m_0$. Since $m_0\in S$, $m_0>a$. However $\frac{m_0+a}{2}\in(a,m_0)$, so $\frac{m_0+a}{2}\in S$ and $\frac{m_0+a}{2}<m_0$. This contradicts that m_0 was the smallest element in S.
- $\inf(S)=a$. a is a lower bound for S. Suppose $m_0>a$ was another lower bound. Since $b\in S$, we have $m_0\in(a,b]$. Furthermore, since $\frac{m_0+a}{2}\in(a,m_0)$, we have $\frac{m_0+a}{2}\in S$ and $\frac{m_0+a}{2}<m_0$.

This contradicts that m_0 was a lower bound of S.

(8) By the Archimedean Property, if $x>0$ and $y>0$, then there exists $n \in \mathbb{N}$ so that $nx>y$

Taking $x=1$ and $y=a$ gives that there exists $n_1 \in \mathbb{N}$ so that $n_1 > a$.

Taking $x=a$ and $y=1$ gives that there exists $n_2 \in \mathbb{N}$ so that $n_2 a > 1 \Leftrightarrow a > \frac{1}{n_2}$.

Let $n = \max\{n_1, n_2\}$. Then $n \in \mathbb{N}$ and $\frac{1}{n} \leq \frac{1}{n_2} < a < n_1 < n$, which gives the result.

(9) Suppose for the sake of contradiction that $a>b$. Then if we define $y=a-b$, $y>0$, and by Q9, there exists $n \in \mathbb{N}$ so that $\frac{1}{n} < y = a-b$. This implies that there exists $n \in \mathbb{N}$ so that $b + \frac{1}{n} < a$, which is a contradiction. Therefore, we must have $a \leq b$.

8

a) A sequence s_n does not converge if, for all $s \in \mathbb{R}$, $\exists\ \varepsilon > 0$ s.t. $\forall N \in \mathbb{R}$ $\exists\ n > N$ s.t. $|s_n - s| > \varepsilon$.

b) i) Fix $\varepsilon > 0$. Note that
$$|s_n - 0| = \left|\frac{1}{n}\sin(2n)\right| = \frac{1}{n}|\sin(2n)| \le \frac{1}{n} < \varepsilon$$
if $n > \frac{1}{\varepsilon}$. Thus, for $N = \frac{1}{\varepsilon}$, we have that, for all $n > N$, $|s_n - 0| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim s_n = 0$.

ii) Fix $\varepsilon > 0$. Note that
$$|d_n - \tfrac{3}{5}| = \left|\frac{3n+4}{5n+3} - \frac{3}{5}\right| = \left|\frac{(3n+4)5 - 3(5n+3)}{5(5n+3)}\right|$$
$$= \left|\frac{11}{5(5n+3)}\right| = \frac{11}{5}\,\frac{1}{5n+3} \le \frac{11}{25}\cdot\frac{1}{n} \le \frac{1}{2n} < \varepsilon$$
if $n > \frac{1}{2\varepsilon}$. Thus, for $N = \frac{1}{2\varepsilon}$, we have that, for all $n > N$, $|d_n - \frac{3}{5}| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim d_n = \frac{3}{5}$.

iii) Fix $s \in \mathbb{R}$ arbitrary. Let $\varepsilon = 2$. Suppose $\exists\ N \in \mathbb{R}$ s.t. $\forall\ n > N$, $|a_n - s| < \varepsilon$.

Note that

$$|a_n - s| = |2\cos(n\pi) - s| < \varepsilon \iff$$
$$s - \varepsilon < 2\cos(n\pi) < s + \varepsilon \iff$$
$$s - 2 < 2\cos(n\pi) < s + 2.$$

In particular, for $n > N$ even, we have $2 < s + 2 \Rightarrow s > 0$. Similarly, for $n > N$ odd, we have $s - 2 < -2 \Rightarrow s < 0$. Since it is impossible to have both $s > 0$ and $s < 0$, we obtain a contradiction.

Thus $\forall N \in \mathbb{R}$, $\exists n > N$ s.t. $|a_n - s| \geq \varepsilon$. This shows a_n does not conver .

10

(a) A sequence s_n converges to a limit $s \in \mathbb{R}$ if, $\forall \varepsilon > 0$, $\exists N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon$.

(b) A sequence s_n does not converge to a limit $s \in \mathbb{R}$ if $\exists \varepsilon > 0$ s.t. for all $N \in \mathbb{R}$, $\exists n > N$ s.t. $|s_n - s| \geq \varepsilon$.

(c) Fix $\varepsilon > 0$. Let $N = \frac{4}{\varepsilon}$. Then $n > N$ ensures

$$\frac{4}{n} < \varepsilon \iff \frac{4n}{n^2} < \varepsilon \iff \frac{n+3n}{n^2} < \varepsilon \implies \frac{n+3}{n^2} < \varepsilon \implies$$

$$\frac{|n-3|}{n^2} < \varepsilon \implies \left|\frac{n-3}{n^2}\right| < \varepsilon \implies \left|\frac{n-3}{n^2+9}\right| < \varepsilon \iff \left|\frac{n-3}{n^2+9} - 0\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this gives the result.

(d) Assume, for the sake of contradiction, that s_n converges to some $s \in \mathbb{R}$. Then, for $\varepsilon = 1$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures

$$\begin{aligned} |s_n - s| < 1 &\iff s-1 < s_n < s+1 \\ &\iff s-1 < (n+1)^2 - 2 < s+1 \\ &\iff s+1 < (n+1)^2 < s+3 \end{aligned}$$

By the lemma following the Archimedean Property, $\exists\ m \in \mathbb{N}$ so that $m > s+3$. Let $k = \max(m, N+1)$. Then $k \geq m > s+3$ (★) and $k > N$. The latter ensures:

$(k+1)^2 < s+3 \Rightarrow k < k^2+2k+1 < s+3$

This contradicts ⊛. Thus s_n must not converge to any $s \in \mathbb{R}$.

11)

(a)

We will show a_n converges to $a = 7/3$. Fix $\epsilon > 0$. Note that

$$|a_n - a| = \left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| = \left|\frac{21n-57}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}\right|$$
$$= \left|\frac{-106}{3(3n+7)}\right| < \epsilon$$

if and only if

$$\frac{108}{3\epsilon} < 3n+7 \Longleftarrow \frac{108}{3\epsilon} < 3n \Longleftrightarrow \frac{108}{\epsilon} < n.$$

Thus, if $N = \frac{108}{\epsilon}$, for all $n > N$, we have $|a_n - a| < \epsilon$. Since ϵ was arbitrary, this shows $\lim_{n\to+\infty} a_n = a$.

(b)

We will show b_n does not converge. Note that the elements in the sequence b_n are $(\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{1}{2}, 0, \dots)$, repeating in this way. Assume, for the sake of contradiction, that b_n converges to some $b \in \mathbb{R}$. Then for $\epsilon = \frac{1}{4} > 0$, there exists N so that $n > N$ ensures

$$|b_n - b| < \epsilon \Longleftrightarrow b - \epsilon < b_n < b + \epsilon \Longleftrightarrow b - \frac{1}{4} < b_n < b + \frac{1}{4}.$$

Since there are infinitely many $n > N$ for which $b_n = -\frac{1}{2}$, we see that

$$b - \frac{1}{4} < -\frac{1}{2} \Longrightarrow b < \frac{-1}{4}.$$

Likewise, since there are infinitely many $n > N$ for which $b_n = \frac{1}{2}$, we see that

$$\frac{1}{2} < b + \frac{1}{4} \Longrightarrow \frac{1}{4} < b.$$

It is impossible to have both $b < \frac{-1}{4}$ and $\frac{1}{4} < b$. Thus, we have found a contradiction. This shows b_n does not converge.