Math 117: Homework 4

Due Friday, February 2nd

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Consider a sequences s_n. Prove that s_n is a bounded sequence if and only if $S = \{s_n : n \in \mathbb{N}\}$ is a bounded set of real numbers.

Question 2

(a) Give an example of two sequences s_n and t_n for which

$$\lim(s_n t_n) \neq (\lim s_n)(\lim t_n).$$

Justify your answer with a proof. In particular, if you claim that the limit of s_n does not exist, you must prove it.

(b) What hypotheses from the theorem that the limit of the product is the product of the limits do not hold in the example you gave in part (a)?

Question 3*

An important lemma in the analysis of sequences is known as the *squeeze lemma*.

LEMMA 1 (Squeeze Lemma)**.** *Consider three sequences a_n, b_n, and s_n. If $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and there exists $s \in \mathbb{R}$ so that*

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = s,$$

then $\lim_{n\to+\infty} s_n = s$.

In this question, you will prove the squeeze lemma and consider an important consequene of this lemma.

(a) Prove the squeeze lemma.

(b) Suppose s_n and t_n are sequences such that $|s_n| \leq t_n$ for all $n \in \mathbb{N}$. Prove that, if $\lim_{n\to+\infty} t_n = 0$, then $\lim_{n\to+\infty} s_n = 0$.

(c) Is the converse to part (b) true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 4

Recall that the *reverse triangle inequality* ensures that, for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$.

Suppose t_n is a convergent sequence, with $\lim_{n\to+\infty} t_n = t$.

(a) Use the definition of convergence to prove that

$$\lim_{n\to+\infty} |t_n| = |t|.$$

(b) As a consequence of part (a), you have proved the following statement:

If t_n is a convergent sequence, then $|t_n|$ is a convergent sequence.

Is the converse true? If so, prove it. If not, give a counterexample and justify your counterexample.

Question 5*

Suppose the limits of the sequences s_n and t_n exist and $a \in \mathbb{R}$.

(a) Suppose $\lim_{n\to\infty} s_n < a$. Prove that $s_n \geq a$ for at most finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : s_n \geq a\}$ is finite.

(b) Suppose $\lim_{n\to+\infty} t_n > 0$. Prove that there exists $b > 0$ so that $t_n \geq b$ for all but finitely many n—in other words, prove that the set $\{n \in \mathbb{N} : t_n < b\}$ is finite.

(c) Suppose $\lim_{n\to+\infty} s_n t_n > 0$. If $\lim_{n\to+\infty} t_n > 0$, prove that $s_n \leq 0$ for at most finitely many $n \in \mathbb{N}$—in other words, prove that the set $\{n \in \mathbb{N} : s_n \leq 0\}$ has finitely many elements.

Question 6*

Suppose that s_n is a convergent sequence with domain $n \in \{1, 2, 3, \dots, \} = \mathbb{N}$.

(a) Fix $m \in \mathbb{N}$ and let t_n be a sequence with domain $n \in \{m, m+1, m+2, \dots\}$ defined by

$$t_n = s_n \text{ for all } n \in \{m, m+1, m+2, \dots\}.$$

Prove that $\lim t_n = \lim s_n$.

(b) Fix $m \in \mathbb{N}$ and let t_n be a sequence with domain $n \in \mathbb{N}$ defined by

$$t_n = s_{n+m} \text{ for all } n \in \mathbb{N}.$$

Prove that $\lim t_n = \lim s_n$.

The moral of part (a) is that the domain of a sequence doesn't affect its limit—all that matters are the values of the sequence and the order in which the values appear. **The moral of part (b)** is that, if you chop off the first m elements of a sequence, it doesn't change the limit.

Question 7

Consider a sequence s_n satisfying $s_n \neq 0$ for all n and for which $\left|\frac{s_{n+1}}{s_n}\right|$ converges to L. If $L < 1$, show that $\lim s_n = 0$.

Hint: Explain why you can select a so that $L < a < 1$ and prove that there exists N so that $n > N$ ensures $|s_{n+1}| < a|s_n|$. Then use induction to show $|s_n| \leq a^{n-N-1}|s_{N+1}|$ for all $n > N$.

Question 8

The following inequality is true for all $p \geq 1$ and $x, y \geq 0$:

$$|x^p - y^p| \leq p\Big(\max\{x^{p-1}, y^{p-1}\}\Big)|x - y|.$$

Suppose t_n is a convergent sequence with $t_n \geq 0$ for all n. Use the above inequality to prove that for any $p \geq 1$,

$$\lim (t_n)^p = (\lim t_n)^p\,.$$

Question 9*

Which of the following is a correct version of the definition of convergence?

A sequence s_n converges to a limit s if...

(a) there exists $\epsilon > 0$ so that, for all $N \in \mathbb{R}$, $n > N$ ensures $|s_n - s| < \epsilon$.

(b) for all $\epsilon \geq 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.

(c) for all $\epsilon > 0$, $|s_n - s| < \epsilon$ for all $n \in \mathbb{N}$.

(d) given $\epsilon > 0$, there is some $N \in \mathbb{R}$ so that $|s_n - s| < \epsilon$ for all $n > N$.

For each of the statements above that are *not* the correct definition of convergence, give an example of either

- a sequence that satisfies the statement but does not converge or
- a sequence that converges but does not satisfy the statement.

In each case, the existence of such an example will illustrate why the statement is *not* equivalent to the correct definition of convergence.

Question 10

Prove the following useful lemma:

LEMMA 2. *If r_n and t_n are sequences whose limits exist and $r_n \leq t_n$ $\forall n \in \mathbb{N}$, then*

$$\lim_{n\to+\infty} r_n \leq \lim_{n\to+\infty} t_n.$$

Do **not** assume that the sequences r_n and t_n converge, only that the limits exist.

Question 11*

Prove that a sequence s_n is increasing if and only if for all $n, m \in \mathbb{N}$, $n \leq m \implies s_n \leq s_m$.

Question 12

Parts (a) and (b) of this question are not related.

(a) Suppose that either $\lim s_n = +\infty$ or $\lim s_n = -\infty$. Prove that s_n does not converge.

(b) Given a sequence s_n, prove that $\lim_{n\to+\infty} s_n = +\infty$ if and only if $\lim_{n\to+\infty} -s_n = -\infty$. (You may use the direction that we already proved in lecture in your proof.)

Part (a) of this question justifies our terminology "diverges to $\pm\infty$" for this type of sequence.

Question 13

Suppose the limit of s_n exists and $k \in \mathbb{R}$. Then $\lim_{n\to+\infty} ks_n = k \lim_{n\to+\infty} s_n$.