Homework 4 Solutions

① First, suppose s_n is a bounded sequence. Then $\exists$ $M \geq 0$ s.t. $|s_n| \leq M$ $\forall$ $n \in \mathbb{N}$, so $-M \leq s_n \leq M$ $\forall$ $n \in \mathbb{N}$. This shows $-M$ is a lower bound for S and M is an upper bound for S. Thus S is a bounded set.

Next, suppose S is a bounded set. Then S is bounded above and below, so $\exists$ $m, M \in \mathbb{R}$ s.t. $m \leq s_n \leq M$ $\forall$ $n \in \mathbb{N}$. Let $L = \max\{|m|, |M|\}$. Then

$$-L \leq -|m| \leq m \leq s_n \leq M \leq |M| \leq L \quad \forall n \in \mathbb{N},$$

so $|s_n| \leq L$ $\forall$ $n \in \mathbb{N}$. This shows s_n is a bounded sequence.

② ⓐ

Let $s_n = (-1)^n$, $t_n = (-1)^n$. Then $s_n t_n = 1$ for all $n \in \mathbb{N}$. As shown in class, neither the limit of s_n nor the limit of t_n exists. Thus, $\lim_{n \to \infty} s_n t_n = 1 \neq (\lim s_n)(\lim t_n)$

(b) Both sequences s_n and t_n must converge for the limit of the product to be the product of the limits.

Q3) ⓐ Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$, for all $\varepsilon > 0$, there exists N_a and N_b so that $n > N_a$ ensures $|a_n - s| < \varepsilon$ and $n > N_b$ ensures $|b_n - s| < \varepsilon$. Recall that $|a_n - s| < \varepsilon \iff s - \varepsilon < a_n < s + \varepsilon$ and $|b_n - s| < \varepsilon \iff s - \varepsilon < b_n < s + \varepsilon$. Define $N = \max\{N_a, N_b\}$. Then for all $\varepsilon > 0$, $n > N$ ensures $s - \varepsilon < a_n$ and $b_n < s + \varepsilon$. Since $a_n \leq s_n \leq b_n$, this shows that for all $\varepsilon > 0$, $n > N$ ensures $s - \varepsilon < s_n < s + \varepsilon$ $\iff |s_n - s| < \varepsilon$, i.e. $\lim_{n\to\infty} s_n = s$.

ⓑ Apply the squeeze lemma, since

$$-t_n \leq s_n \leq t_n$$

and $\lim_{n\to\infty} t_n = 0$, $\lim_{n\to\infty} -t_n = (-1)(0) = 0$.

ⓒ If $\lim_{n\to\infty} s_n = 0$, it is not necessarily true that $\lim_{n\to\infty} t_n = 0$. For example, if $s_n = 0$ $\forall n \in \mathbb{N}$ and $t_n = 1$ $\forall n \in \mathbb{N}$, then $|s_n| \leq t_n$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = 0$, but $\lim_{n\to\infty} t_n = 1 \neq 0$.

4

ⓐ Fix $\varepsilon > 0$. Note that, by the reverse triangle inequality,

$$\big||t| - |t_n|\big| \leq |t - t_n|.$$

Since $\lim_{n\to\infty} t_n = t$, $\exists\ N \in \mathbb{R}$ s.t. $n > N$ ensures $|t - t_n| < \varepsilon$, hence $\big||t| - |t_n|\big| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim |t_n| = |t|$.

ⓑ The converse is not true.
Let $t_n = (-1)^n$. Then $|t_n| = 1$ is a convergent sequence, but t_n is not a convergent sequence.

5 ⓐ First, suppose $\lim_{n\to\infty} s_n = s$ for $s < a$. If we define $\varepsilon = a - s$, then $\varepsilon > 0$.
By definition of convergence, there exists $N \in \mathbb{R}$ s.t. $n > N$ ensures $|s_n - s| < \varepsilon \iff s - \varepsilon < s_n < s + \varepsilon$

Thus $n > N$ ensures $s_n < s + \varepsilon = a$.
Hence $\{n \in \mathbb{N} : s_n \geq a\} \subseteq \{1, 2, \dots, \lfloor N \rfloor\}$.
Thus, $\{n \in \mathbb{N} : s_n \geq a\}$ is a finite set.

Next, suppose $\lim_{n \to \infty} s_n = -\infty$. Let $M = \min\{-1, a\}$. Then $\exists$ N s.t. $s_n < M \leq a$ for all $n > N$.
Thus, $\{n \in \mathbb{N} : s_n \geq a\}$ is a finite set.

(b) First, suppose $\lim_{n \to \infty} t_n = t > 0$. Then $\lim_{n \to \infty} t_n > \frac{t}{2}$. Applying part (a) with $s_n = -t_n$ and $a = -\frac{t}{2}$, we obtain that
$\{n \in \mathbb{N} : s_n \geq a\} = \{n \in \mathbb{N} : -t_n \geq -\frac{t}{2}\}$
$= \{n \in \mathbb{N} : t_n \leq \frac{t}{2}\} \supseteq \{n \in \mathbb{N} : t_n < \frac{t}{2}\}$
are all finite sets. This shows the result for $b = \frac{t}{2}$.

Now, suppose $\lim_{n \to \infty} t_n = +\infty$. Then $\exists$ N s.t. $\forall n > N$, $t_n > 1$. Thus,
$\{n \in \mathbb{N} : t_n \leq 1\} \supseteq \{n \in \mathbb{N} : t_n < 1\}$ is finite. This shows the result for $b = 1$.

(c) By part (b), $\exists$ $b_1 > 0$ so that $t_n \geq b_1$

for all but finitely many n and $\exists\, b_2 > 0$ so that $s_n t_n \geq b_2$ for all but finitely many n. Thus, there exists N_1, N_2 so that $n > N_1$ ensures $t_n \geq b_1 \geq 0$ and $n > N_2$ ensures $t_n s_n \geq b_2 \geq 0$. Thus $n > \max\{N_1, N_2\}$ ensures $s_n \geq 0$. This shows $\{n \in \mathbb{N} : s_n \leq 0\}$ has at most $\max\{N_1, N_2\}$ elements.

(6) Suppose $\lim_{n\to\infty} s_n = s$

(a) Fix $\varepsilon > 0$. Since s_n converges to s, there exists N s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

Let $\tilde{N} = \max\{N, m\}$. Then $n > \tilde{N}$ ensures

$$|t_n - s| = |s_n - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} t_n = s = \lim_{n\to\infty} s_n$.

(b) Fix $\varepsilon > 0$. Since s_n converges to s, there exists N s.t. $n > N$ ensures

$$|s_n - s| < \varepsilon.$$

Since $n > N$ implies $n+m > N$, we have

$$|t_n - s| = |s_{n+m} - s| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} t_n = s = \lim_{n\to\infty} s_n$.

⑦ Since $L<1$, if we define $a=\frac{L+1}{2}$, then $L<a<1$. By Q6ⓐ, $\{n\in\mathbb{N} : |\frac{s_{n+1}}{s_n}|\geq a\}$ is finite. Thus, there exists N s.t. $n>N$ ensures $|\frac{s_{n+1}}{s_n}|<a$, or equivalently, $\underbrace{|s_{n+1}|<a|s_n|}_{\star}$.

We now prove that $|s_n|\leq a^{n-N-1}|s_{N+1}|$ for all $n>N$ using induction. For the base case of $n=N+1$, note that $|s_{N+1}|=a^0|s_{N+1}|=a^{n-N-1}|s_{N+1}|$. For the inductive step, assume $|s_n|\leq a^{n-N-1}|s_{N+1}|$. By ($\star$) above, this implies $|s_{n+1}|\leq a|s_n|\leq a^{n+1-N-1}$, which completes the proof of the inductive step.

Finally, we use that $|s_n|\leq a^{n-N-1}|s_{N+1}|$ to prove that $\lim_{n\to\infty} s_n=0$. By Question ③ⓑ, it suffices to show $\lim_{n\to\infty} a^{n-N-1}|s_{N+1}|=0$. Since $a<1$, $\lim_{n\to\infty} a^n=0$. Thus, using that the limit of the product is the product of the limits, we obtain $\lim_{n\to\infty} a^{n-N-1}|s_{N+1}|=\left(\lim_{n\to\infty} a^n\right)a^{-N-1}|s_{N+1}|=0$.

(8) By the inequality,

$$|t_n^p - t^p| \le p \max\{(t_n)^{p-1}, (t)^{p-1}\} |t_n - t|$$

Since t_n is a convergent sequence, it is bounded and $\exists\ M>0$ s.t. $|t_n|<M$ for all n. By Question 5, $\lim_{n\to\infty} |t_n| = |t|$. Let $\tilde{M} = \max(M, |t|)$. Then the above inequality ensures.

$$(*)\ |t_n^p - t^p| \le p\, \tilde{M}^{p-1} |t_n - t|.$$

Fix $\varepsilon>0$. Since t_n converges to t, $\exists\ N$ s.t. $n>N$ ensures $|t_n - t| < \frac{\varepsilon}{p\tilde{M}^{p-1}}$.

Then, by $(*)$, $n>N$ ensures $|t_n^p - t^p| < \varepsilon$. Since $\varepsilon>0$ was arbitrary, this shows $\lim t_n^p = t^p$.

(9) The correct definition is (d).

(a) Consider $s_n = (1,-1,1,-1,\dots)$, $s=0$. Then for $\varepsilon = 2$ and all $N \in \mathbb{R}$, $n>N$ ensures $|s_n - s| < \varepsilon$.

(b) Consider $s_n = \frac{1}{n}$, $s = 0$. For $\varepsilon = 0$ there is no $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - 0| < \varepsilon$.

(c) Consider $s_n = \frac{1}{n}$, $s = 0$. For $\varepsilon = \frac{1}{4}$, $|s_n - s| < \varepsilon$ is not true for all $n \in \mathbb{N}$.

(10) If $\lim_{n\to\infty} r_n = -\infty$, the result is immediate. Thus, it remains to consider the remaining cases.

Case 1: Suppose $\lim_{n\to\infty} r_n = r \in \mathbb{R}$

Case 1a: If $\lim_{n\to\infty} t_n = +\infty$, we are done.

Case 1b: Suppose $\lim_{n\to\infty} t_n = t \in \mathbb{R}$. Assume for the sake of contradiction that $t < r$. Let $\varepsilon = \frac{r-t}{2} > 0$. Then $\exists N_r, N_t$ s.t. $n > N_r$ ensures $|r_n - r| < \varepsilon$ and $n > N_t$ ensures $|t_n - t| < \varepsilon$. Let $N = \max\{N_r, N_t\}$. Then $n > N$ ensures

$$t_n < t + \varepsilon = t + \frac{r-t}{2} = \frac{t+r}{2} = r - \frac{r-t}{2} = r - \varepsilon < r_n.$$

This contradicts that $r_n \leq t_n \ \forall n \in \mathbb{N}$. Thus $\lim_{n\to\infty} t_n = t \geq r = \lim_{n\to\infty} r_n$.

Case 1c: Suppose $\lim_{n\to\infty} t_n = -\infty$. Then $\exists N_t$ s.t. $\forall n > N_t$, $t_n < r - 1$. There also exists N_r s.t. $\forall n > N_t$, $r - 1 < r_n$. Thus for $N = \max\{N_t, N_r\}$, $n > N$ ensures

$t_n < r-1 < r_n$.

Again, this contradicts that $r_n \le t_n \ \forall n \in \mathbb{N}$.
Thus $\lim_{n\to\infty} t_n = -\infty$ is impossible.

Case 2: Suppose $\lim_{n\to\infty} r_n = +\infty$. Fix $M>0$. Then $\exists\ N$ s.t. $\forall n \ge N$, $M < r_n \le t_n$. This shows $\lim_{n\to\infty} t_n = +\infty$.

(11) Suppose s_n is increasing. Fix $n \in \mathbb{N}$.
We will prove

$$m \ge n \Rightarrow s_m \ge s_n$$

by induction.

Base case: $m=n$. By definition $s_m = s_n$.

Inductive step: Suppose $m \ge n$ and $s_m \ge s_n$. Since it is an increasing sequence, $s_{m+1} \ge s_m \ge s_n$. This shows the inductive step.

Now, suppose $m \ge n \Rightarrow s_m \ge s_n \ \forall n, m \in \mathbb{N}$. Take $m = n+1$. Then $s_{n+1} \ge s_n \ \forall n \in \mathbb{N}$. This shows s_n is increasing.

(12) (a) Assume for the sake of contradiction that s_n converges to some $s \in \mathbb{R}$. Then $\exists N$ s.t. $\forall n > N$, $s-1 < s_n < s+1$. Thus, it is impossible for s_n to diverge to $+\infty$

since, for $M = |s+1|$, there is no N_M s.t. $s_n > M = |s+1| \geq s+1$ $\forall n > N_M$. Likewise, it is impossible for s_n to diverge to $-\infty$ since for $m = -|s-1|$, there is no $\tilde{N}_m$ s.t. $s_n < m = -|s-1| \leq s-1$ $\forall n > \tilde{N}_m$.

(b) Suppose $\lim_{n\to\infty} s_n = +\infty$. Fix $m < 0$. Then $-m > 0$, so $\exists$ N s.t. $n > N$ ensures $s_n > -m \Rightarrow m > -s_n$. Thus $\lim_{n\to\infty} -s_n = -\infty$.

Now, suppose $\lim_{n\to\infty} -s_n = -\infty$. Fix $M > 0$. Then $-M < 0$, so $\exists$ N s.t. $n > N$ ensures $-s_n < -M \Rightarrow s_n > M$. Thus $\lim_{n\to\infty} s_n = +\infty$.

13)

Case 1: $\lim_{n\to+\infty} s_n \in \mathbb{R}$

Since $t_n = (k, k, k, \ldots)$ is a sequence that converges to k and s_n is a convergent sequence, by the theorem that the limit of the product is the product of the limits,

$$\lim_{n\to+\infty} ks_n = \lim_{n\to+\infty} t_n s_n = \left(\lim_{n\to+\infty} t_n\right)\left(\lim_{n\to+\infty} s_n\right) = k \lim_{n\to+\infty} s_n.$$

Case 2:$\lim_{n\to+\infty} s_n = \pm\infty$ and $k = 0$

Then $ks_n = (0, 0, 0, \ldots)$ converges to $0 = k \cdot (+\infty) = k \lim_{n\to+\infty} s_n$.

Case 3a: $\lim_{n\to+\infty=+\infty}$ and $k > 0$

We must show that ks_n diverges to $+\infty$. Fix $M > 0$. Since s_n diverges to ∞, there exists N so that $n >$ ensures $s_n > M/k \implies ks_n > M$. This shows $\lim_{n\to+\infty} ks_n = +\infty$.

Case 3b: $\lim_{n\to+\infty=+\infty}$ and $k < 0$

Then $-(ks_n) = (-k)s_n$. By Case 3a, $\lim_{n\to+\infty}(-k)s_n = +\infty$. By Q12(b), this implies $\lim_{n\to+\infty} ks_n = -\infty$.

Case 4a: $\lim_{n\to+\infty} s_n = -\infty$ and $k > 0$

Then $-(ks_n) = k(-s_n)$. By Q12(b), $\lim_{n\to+\infty} -s_n = +\infty$. Thus, Case 3a ensures $\lim_{n\to+\infty} k(-s_n) = +\infty$. Thus, by Q12 again, $\lim_{n\to+\infty} ks_n = -\infty$.

Case 4b: $\lim_{n\to+\infty} s_n = -\infty$ and $k < 0$

Then $-(ks_n) = (-k)s_n$. By Case 4a, $\lim_{n\to+\infty}(-k)s_n = -\infty$. By Q12(b), this implies $\lim_{n\to+\infty} ks_n = +\infty$.