# Math 117: Homework 5

Due Friday, February 16th

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

### Question 1

In HW4, Q7, you considered a sequence $s_n$ satisfying $s_n \neq 0$ for all $n \in \mathbb{N}$ and for which the limit $\lim_{n\to+\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ exists. You showed that, if $L < 1$,then $\lim s_n = 0$.

Now prove that if $L > 1$ (which includes the possibility $L = +\infty$), then $\lim |s_n| = +\infty$.

**Hint:** Explain why you can apply HW4, Q7, to the sequence $t_n = \frac{1}{|s_n|}$.

### Question 2*

Define a sequence $s_n$ as follows: $s_1 = 1$ and, for $n \geq 1$, $s_{n+1} = \frac{1}{3}(s_n + 1)$.

(a) Use induction to prove that $s_n \geq 1/2$ for all $n$.

(b) Use the definition of the sequence and part (a) to show that the sequence is decreasing.

(c) Prove that $\lim s_n = s$ for some $s \in \mathbb{R}$.

(d) Use HW4 Q6(b) to prove that $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} s_{n+1}$.

(e) Use part (d), the definition of $s_n$, and the limit theorems to find the value of $s$.

### Question 3*

Let $s_n$ be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove $\sigma_n$ is an increasing sequence.

### Question 4

Suppose $\lim t_n > 0$ and $\lim s_n = +\infty$ Prove that $\lim s_n t_n = +\infty$. (Hint: Use HW4, Q5.)

### Question 5

Suppose $a < b + \epsilon$ for all $\epsilon > 0$. Prove that $a \leq b$. (Compare to HW1 Q3 and HW3 Q9.)

### Question 6*

(a) Given a nonempty subset $S \subseteq \mathbb{R}$ and $k > 0$, let $kS = \{ks : s \in S\}$. Prove that $\sup(kS) = k\sup(S)$. You may use the fact that, for any $k > 0$, $k \cdot (+\infty) = +\infty$ and $(-k) \cdot (+\infty) = -\infty$.

(b) For $k > 0$, prove that $\limsup(ks_n) = k \limsup s_n$.

(c) How does the result of part (a) change when $k < 0$? Prove it.

## Question 7*

In class, we defined that a sequence $s_n$ of real numbers *converges* to some $s \in \mathbb{R}$ provided that, for all $\epsilon > 0$, there exists $N \in \mathbb{R}$ so that $n > N$ ensures $|s_n - s| < \epsilon$.

Prove that the above definition is equivalent to the following definition: *for all* $\epsilon > 0$, *there exists* $N \in \mathbb{R}$ *so that* $n \geq N$ *ensures* $|s_n - s| < \epsilon$.

## Question 8

Prove $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

## Question 9

Given a sequence $s_n$, prove that $\limsup |s_n| < +\infty$ if and only if $s_n$ is bounded.

## Question 10*

Determine whether the following statements are true or false. If they are true, prove them. If they are false, give a counterexample and justify it.

(a) If a sequence $s_n$ satisfies $\limsup s_n = 2$, then $s_n > 1.99$ for all $n$ large enough.

(b) If a sequence $s_n$ satisfies $\limsup s_n = b$, then $s_n \leq b$ for all $n$ large enough.

## Question 11*

Give examples of...

(a) A sequence $x_n$ of irrational numbers having a limit that is a rational number.

(b) A sequence $r_n$ of rational numbers having a limit that is an irrational number.

Justify your answers.

## Question 12*

Let $s_n$ and $t_n$ be the following sequences that repeat in cycles of four:

$$s_n = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \dots)$$
$$t_n = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \dots).$$

Compute the following quantities. You do not need to justify your answer.

(a) $\liminf s_n$

(b) $\limsup s_n$

(c) $\liminf t_n$

(d) $\limsup t_n$

(e) $\liminf s_n + \liminf t_n$

(f) $\liminf s_n + \limsup t_n$

(g) $\limsup s_n + \limsup t_n$

(h) $\limsup(s_n t_n)$

(i) $\liminf(s_n + t_n)$

(j) $\limsup(s_n + t_n)$

(k) $\liminf(s_n t_n)$

Pay attention to the above examples where we see that the liminf and limsup of the sum/product is not necessarily equal to the sum/product of the liminf and limsup.