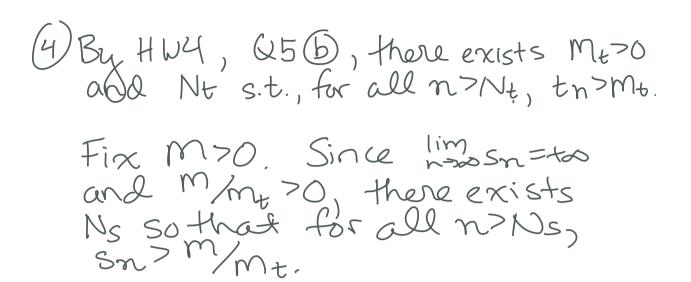
Homework 5 Solutions C Katy Craig, 2024 1) Suppose L>1. Define tn= 1/1sn1, and note that lim  $\left|\frac{tn+1}{tn}\right| = \lim \frac{tn+1}{tn} = \lim \frac{lsnl}{lsn+1} \lim \frac{l}{lsn+1}$ For the sequence Isntil/Isn1, the assumption that sn =0 + n ensures the denominator is not zoro. We also know that the denominator converges and its limit is L>1, hence nonzero. Therefore, by the fact that the limit of a quotient is the quotient of the limits,  $\lim_{t \to 1} |\frac{tnrt}{t}| = \frac{1}{2}$ . Since by definition tn 70 4n, and the fact that L> ( ensures lim [tnt] <1, by HW4, Q7, we conclude that lim tn=0. Since tn is a sequence of positive numbers, by the theorem from class, lim to the lim Isn I=+00. a) Base case: When n=1,  $s_1=12\frac{1}{2}$ . Inductive step: Suppose sn= 2. We aim to Show  $S_{n+1} = \frac{1}{2}$ . By definition  $S_{n+1} = \frac{1}{3}(S_n + 1)$ . Since  $S_n = \frac{1}{2}$ ,  $S_n + 1 = \frac{1}{3}$ , so  $S_{n+1} = \frac{1}{3}(S_n + 1) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}$ .

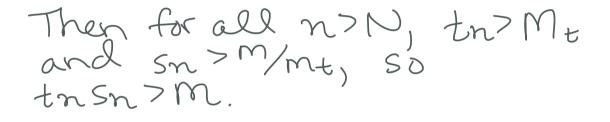
This completes the proof. (b) We aim to show  $Snti \leq Sn$  for all nt/N. By part (a),  $sn^2 \leq 1$ ,  $so = \frac{2}{3}sn^2 \leq 1$ . Thus, by definition of the sequence,  $S_{n+1} = \frac{1}{3}(S_n+1) = \frac{S_n}{3} + \frac{1}{3} = \frac{S_n}{3} + \frac{2S_n}{3} = S_n,$ which completes the proof. (C) Since sn is a decreasing sequence, SIZSN VNEIN. Since Snzuz VneiN, we have  $\frac{1}{2} \leq sn \leq s_1 = 1$  ynen. Thus sn is a bounded, decreasing sequence. Since all bounded monotone Sequences converge, limas sn=s for some SER. (d) This is an immediate consequence HW4 (6)(b) for m=1. (e) By part (d) and the limit theorems,  $S = \lim_{n \to \infty} S_{n+1} = \lim_{n \to \infty} \frac{1}{3}(S_n + 1) = \frac{1}{3}(S + 1).$ 

Thus, 
$$\frac{2}{3}s = \frac{1}{3}$$
, so  $s = \frac{1}{2}$ .

(3) We must show  $s_{n+1} = \frac{1}{n+1} (s_1 + s_2 + ... + s_{n+1})^2 \frac{1}{n} (s_1 + s_2 + ... + s_n) = s_n$ which is equivalent to showing  $(s_1 + s_2 + ... + s_{n+1})^2 \frac{n+1}{n} (s_1 + s_2 + ... + s_n) = (1 + \frac{1}{n} (s_1 + ... + s_n))$ .  $= s_1 + ... + s_n + \frac{1}{n} (s_1 + ... + s_n)$ . Subtracting  $s_1 + ... + s_n$  from both sides show this is equivalent to showing  $s_{n+1} \ge \frac{1}{n} (s_1 + ... + s_n)$ . Multiplying both sides by  $n_1$  this is equivalent to  $n \cdot s_{n+1} \ge s_1 + ... + s_n$ . Since  $s_n$  is increasing,  $s_{n+1} \ge s_i + ... + s_n$ .



Take N=max(N+,Ns).



Since M>O was arbitrary, this shows lim the shows ()

(5) Assume, for the sake of contradiction, that a>b. Define E=a-b>0. Then b+E=b+(a-b)=a. This contradicts our assumption that a < b + E for all E>0. Thus, we must have  $a \le b$ .

(a) Case 1: 
$$sup(s) = 1 \text{ os. Assume, for the sake of contradiction, that  $M$  is an upper bound (for kS. Then  $s \leq \frac{m}{k}$  bsts, which is a contradiction. Thus,  $sup(kS) = +\infty$ .$$

$$(ax 2: sup(s) \in \mathbb{R}$$
. Since  $sup(s)$  is an upper  
bound for S,  $s = sup(s)$   $\forall s \in S = > ks = ksup(s)$   
 $\forall s \in S = > ksup(s)$  is an upper bound for  $kS$ .  
Furthermore if Mis another upper bound of  $kS$ ,  
then  $\frac{m}{k}$  is an upper bound of S, so  $sup(s) \leq \frac{m}{k}$   
 $=> ksup(s) \leq M$ . This shows  $ksup(s)$  is the least  
upper bound of  $kS$ .

 $\bigcirc If C<0, -C>0. Thus, for any T=R inf(CT) = - sup(-CT) = -(-C) sup(T) = C sup(T).$ Thus dinf(cT) = sup(T). Fix k<0 and S=R. Let<sup>40</sup>T = kS and c=k. Then S= tr. Thus,  $sup(kS) = sup(T) \stackrel{\text{\tiny def}}{=} kinf(kT) = kinf(S).$ 

(7) Call the first definition DEF1 and the second DEF2. Suppose sn converges by DEF1. Fix  $\varepsilon > 0$ . DEF1 ensures  $\exists N \text{ s.t. } n > N \Longrightarrow |\text{sn-sl} < \varepsilon$ . Let  $\tilde{N} = N + 1$ . Then  $n \ge \tilde{N} \Longrightarrow |\text{sn-sl} < \varepsilon$ . Thus, sn converges by DEF2.

Next, suppose on converges by DEF2. Fix E>O. DEF2 ensures Z N s.t. nZN => Isn-sl<E. Thus n>N=> Isn-sl<E. Thus, Sn converges by DEF1.

(8) First, suppose limsn=0. By HW3, Q4, limlsn=0, so limsup Isnl=liming Isnl=limlsn=0. Now suppose limsup IsnI=O. By Definition, this implies him an = 0, where an=sup{Isnl:n=NJ. Fix ==0; and choose Noso that N>No ensures lan-01<EE>lanke. (=) an < 2, since an is nonnegative. In particular, aNoti < E, so by definition of ano, we have that n>Not l'ensures Bulke. Therefore how sn=0. (9) Assume sn is a bounded sequence. Then 7 Mo s.t. Isn1=Mo YONEN. Hence supElsnl:n>NJ=Mo V NEN. By HW4, Q1, 1, mos sup €lsn1:n>N} ∈ Mo, so ) limsup |snl & Mo < tos.

Now, assume limsup Isn 2-100. Recall that limsup Isn = limsup Isn 2-100. Recall that Since and is a convergent sequence, it is bounded, and I mo s.t. land = Mo & NE/N. In particular, la1 = Mo & Isup Isn 1-2] = Mo,

so I snl = max {Isil, Mo}. Thus in  
is a bounded sequence.  
(D) (D) False. Consider: Sn = (-1)<sup>n</sup> 2.  
Then limsup sn = lim sup {sn: n7N}  
= limso 2 = 2.  
However, all odd elements of sn  
are strictly less than 1.99.  
(D) False. Consider sn = b + 1.  
Since Sn is convergent,  
lim sn = b = limsup sn.  
Now for all n.  
(U) (D) Define xn = 
$$\frac{12}{n}$$
. As shown in class,  
IZ is an irrational number. Since the is  
a field, the product of two rational  
humbors is a rational number. Since NEQ  
and xn: n = IZ & the, we must have that  
Xn & Q, so Xn is a sequence of  
irrational number.

Claim: lim xn=0. We must show that

for all E>0, there exists N S,t. n>N ensures Ixn/<E. Note that  $|\chi_n| = |\frac{\sqrt{2}}{n}| = \frac{\sqrt{2}}{n} < \varepsilon \iff \frac{\sqrt{2}}{\varepsilon} < n$ Therefore, for all E>0, if we take N= E, then for all n N, Kn < E.

(b) Define rn=1.41421... first n digite of decimal approximation of the Or more precisely, we define rn by rn=[12.10n]/10n, where Las represents the largest integer less than or equal to a. Then mEQ. Claim: "morn= FZ. Note that  $|r_n - f_z| = |0^{-n}|L^{r_z \cdot 10^n}| - f_z \cdot 10^n| \le |0^{-n}|$ and  $10^{-n} < \varepsilon \iff \frac{1}{\varepsilon} < 10^{n} \iff \log \frac{1}{\varepsilon} < n$ . Therefore, for all 2>0, if we take N=logio =, then for all n>N, 1m-13/28

12)
a) 0
b) 2
c) 0
d) 2
e) 0+0 = 0
f) 0+2 = 2
g) 2+2 = 0
h) 2
i) 1
j) 3