Homework 5 Solutions

① Suppose $L>1$. Define $t_n = \frac{1}{|s_n|}$, and note that $\lim \left|\frac{t_{n+1}}{t_n}\right| = \lim \frac{t_{n+1}}{t_n} = \lim \frac{|s_n|}{|s_{n+1}|} = \lim \frac{1}{|s_{n+1}|/|s_n|}$

For the sequence $\frac{1}{|s_{n+1}|/|s_n|}$, the assumption that $s_n \neq 0 \ \forall n$ ensures the denominator is not zero. We also know that the denominator converges and its limit is $L>1$, hence nonzero. Therefore, by the fact that the limit of a quotient is the quotient of the limits, $\lim \left|\frac{t_{n+1}}{t_n}\right| = \frac{1}{L}$.

Since by definition $t_n \neq 0 \ \forall n$, and the fact that $L>1$ ensures $\lim \left|\frac{t_{n+1}}{t_n}\right| < 1$, by HW4, Q7, we conclude that $\lim t_n = 0$. Since t_n is a sequence of positive numbers, by the theorem from class, $\lim \frac{1}{t_n} = \lim |s_n| = +\infty$.

②

ⓐ Base case: When $n=1$, $s_1 = 1 \geq \frac{1}{2}$.

Inductive step: Suppose $s_n \geq \frac{1}{2}$. We aim to show $s_{n+1} \geq \frac{1}{2}$. By definition $s_{n+1} = \frac{1}{3}(s_n+1)$.

Since $s_n \geq \frac{1}{2}$, $s_n + 1 \geq \frac{3}{2}$, so

$$s_{n+1} = \frac{1}{3}(s_n+1) \geq \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}.$$

This completes the proof.

(b) We aim to show $s_{n+1} \le s_n$ for all $n \in \mathbb{N}$. By part (a), $s_n \ge \frac{1}{2}$, so $\frac{2}{3} s_n \ge \frac{1}{3}$. Thus, by definition of the sequence,

$$s_{n+1} = \frac{1}{3}(s_n + 1) = \frac{s_n}{3} + \frac{1}{3} \overset{(a)}{\le} \frac{s_n}{3} + \frac{2s_n}{3} = s_n,$$

which completes the proof.

(c) Since s_n is a decreasing sequence, $s_1 \ge s_n$ $\forall n \in \mathbb{N}$. Since $s_n \ge \frac{1}{2}$ $\forall n \in \mathbb{N}$, we have $\frac{1}{2} \le s_n \le s_1 = 1$ $\forall n \in \mathbb{N}$. Thus s_n is a bounded, decreasing sequence. Since all bounded monotone sequences converge, $\lim_{n\to\infty} s_n = s$ for some $s \in \mathbb{R}$.

(d) This is an immediate consequence HW4 (6)(b) for $m=1$.

(e) By part (d) and the limit theorems,

$$s = \lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{1}{3}(s_n + 1) = \frac{1}{3}(s+1).$$

Thus, $\frac{2}{3}s = \frac{1}{3}$, so $s = \frac{1}{2}$.

(3) We must show

$\sigma_{n+1} = \frac{1}{n+1}(s_1 + s_2 + \dots + s_{n+1}) \geq \frac{1}{n}(s_1 + s_2 + \dots + s_n) = \sigma_n$,

which is equivalent to showing

$(s_1 + s_2 + \dots + s_{n+1}) \geq \frac{n+1}{n}(s_1 + s_2 + \dots + s_n) = (1 + \frac{1}{n})(s_1 + \dots + s_n)$.

$= s_1 + \dots + s_n + \frac{1}{n}(s_1 + \dots + s_n)$

Subtracting $s_1 + \dots + s_n$ from both sides shows this is equivalent to showing $s_{n+1} \geq \frac{1}{n}(s_1 + \dots + s_n)$. Multiplying both sides by n, this is equivalent to $n s_{n+1} \geq s_1 + \dots + s_n$. Since s_n is increasing, $s_{n+1} \geq s_i \ \forall\, i = 1, \dots, n$ which gives the result.

(4) By HW4, Q5 (b), there exists $M_t > 0$ and N_t s.t., for all $n > N_t$, $t_n > M_t$.

Fix $M > 0$. Since $\lim_{n\to\infty} s_n = +\infty$ and $M/M_t > 0$, there exists N_s so that for all $n > N_s$, $s_n > M/M_t$.

Take $N = \max(N_t, N_s)$.

Then for all $n > N$, $t_n > M_t$ and $s_n > M/M_t$, so $t_n s_n > M$.

Since $M > 0$ was arbitrary, this shows $\lim_{n\to\infty} t_n s_n = +\infty$.

(5) Assume, for the sake of contradiction, that $a > b$. Define $\varepsilon = a - b > 0$. Then $b + \varepsilon = b + (a-b) = a$. This contradicts our assumption that $a < b + \varepsilon$ for all $\varepsilon > 0$. Thus, we must have $a \leq b$.

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a) <u>Case 1</u>: $\sup(S)=+\infty$. Assume, for the sake of contradiction, that m is an upper bound for kS. Then $s \le \frac{m}{k}$ $\forall s\in S$, which is a contradiction. Thus, $\sup(kS)=+\infty$.

<u>Case 2</u>: $\sup(S)\in\mathbb{R}$. Since $\sup(S)$ is an upper bound for S, $s\le \sup(S)$ $\forall s\in S \Rightarrow ks\le k\sup(S)$ $\forall s\in S \Rightarrow k\sup(S)$ is an upper bound for kS. Furthermore if M is another upper bound of kS, then $\frac{m}{k}$ is an upper bound of S, so $\sup(S)\le\frac{m}{k}$ $\Rightarrow k\sup(S)\le m$. This shows $k\sup(S)$ is the least upper bound of kS.

b)
$$\limsup_{n\to\infty}(ks_n) = \lim_{N\to\infty}\sup\{ks_n : n>N\}$$
$$\overset{ⓐ}{=} \lim_{N\to\infty} k\sup\{s_n : n>N\}$$
$$= k\lim_{N\to\infty}\underbrace{\sup\{s_n : n>N\}}_{a_N}$$
$$= k\limsup_{n\to\infty} s_n$$

If a_N converges, this is because limit of product is product of limits. If $\lim a_N=+\infty$, it is Q4. If $\lim a_N=-\infty$, it is Q4 and HW4 Q10.

(c) If $c<0$, $-c>0$. Thus, for any $T \subseteq \mathbb{R}$

$\inf(cT) \overset{HW2, Q7}{=} -\sup(-cT) \overset{part (a)}{=} -(-c)\sup(T) = c\sup(T)$.

Thus $\frac{1}{c}\inf(cT) = \sup(T)$.

Fix $k<0$ and $S \subseteq \mathbb{R}$. Let$^{(*)}$ $T = kS$ and $c = \frac{1}{k}$. Then $S = \frac{1}{k}T$. Thus,

$\sup(kS) = \sup(T) \overset{(*)}{=} k\inf(\frac{1}{k}T) = k\inf(S)$.

(7) Call the first definition DEF 1 and the second DEF 2. Suppose s_n converges by DEF 1. Fix $\varepsilon>0$. DEF 1 ensures $\exists$ N s.t. $n>N \Rightarrow |s_n - s| < \varepsilon$. Let $\tilde{N} = N+1$. Then $n \geq \tilde{N} \Rightarrow |s_n - s| < \varepsilon$. Thus, s_n converges by DEF 2.

Next, suppose s_n converges by DEF 2. Fix $\varepsilon>0$. DEF 2 ensures $\exists$ N s.t. $n \geq N \Rightarrow |s_n - s| < \varepsilon$. Thus $n>N \Rightarrow |s_n - s| < \varepsilon$. Thus, s_n converges by DEF 1.

(8) First, suppose $\lim s_n = 0$. By HW3, Q4, $\lim |s_n| = 0$, so $\limsup |s_n| = \liminf |s_n| = \lim |s_n| = 0$.

Now suppose $\limsup |s_n| = 0$. By definition, this implies $\lim_{N\to\infty} a_N = 0$, where $a_N = \sup\{|s_n| : n > N\}$. Fix $\varepsilon > 0$, and choose N_0 so that $N > N_0$ ensures $|a_N - 0| < \varepsilon \iff |a_N| < \varepsilon$. $\iff a_N < \varepsilon$, since a_N is nonnegative. In particular, $a_{N_0+1} < \varepsilon$, so by definition of a_{N_0}, we have that $n > N_0 + 1$ ensures $|s_n| < \varepsilon$. Therefore $\lim_{n\to\infty} s_n = 0$.

(9) Assume s_n is a bounded sequence. Then $\exists\, M_0$ s.t. $|s_n| \le M_0$ $\forall n \in \mathbb{N}$. Hence $\sup\{|s_n| : n > N\} \le M_0$ $\forall N \in \mathbb{N}$. By HW4, Q1,
$\lim_{N\to\infty} \sup\{|s_n| : n > N\} \le M_0$, so $\limsup_{n\to\infty} |s_n| \le M_0 < +\infty$.

Now, assume $\limsup_{n\to\infty} |s_n| < +\infty$. Recall that

$$\limsup_{n\to\infty} |s_n| = \lim_{N\to\infty} \underbrace{\sup\{|s_n| : n > N\}}_{a_N}.$$

Since a_N is a convergent sequence, it is bounded, and $\exists\, M_0$ s.t. $|a_N| \le M_0$ $\forall N \in \mathbb{N}$. In particular, $|a_1| \le M_0 \iff |\sup\{|s_n| : n > 1\}| \le M_0$.

so $|s_n| \le \max\{|s_1|, M_0\}$. Thus s_n is a bounded sequence.

(10) (a) False. Consider: $s_n = (-1)^n 2$.

Then $\limsup_{n\to\infty} s_n = \lim_{N\to\infty} \sup\{s_n : n > N\}$

$$= \lim_{N\to\infty} 2 = 2.$$

However, all odd elements of s_n are strictly less than 1.99.

(b) False. Consider $s_n = b + \frac{1}{n}$.

Since s_n is convergent, $\lim_{n\to\infty} s_n = b = \limsup_{n\to\infty} s_n$.

However $s_n > b$ for all n.

(11) (a) Define $x_n = \frac{\sqrt{2}}{n}$. As shown in class, $\sqrt{2}$ is an irrational number. Since $\mathbb{Q}$ is a field, the product of two rational numbers is a rational number. Since $\mathbb{N} \subseteq \mathbb{Q}$ and $x_n \cdot n = \sqrt{2} \notin \mathbb{Q}$, we must have that $x_n \notin \mathbb{Q}$, so x_n is a sequence of irrational numbers.

Claim: $\lim_{n\to\infty} x_n = 0$. We must show that

for all $\varepsilon>0$, there exists N s.t. $n>N$ ensures $|x_n|<\varepsilon$. Note that
$|x_n| = |\frac{\sqrt{2}}{n}| = \frac{\sqrt{2}}{n} < \varepsilon \iff \frac{\sqrt{2}}{\varepsilon} < n$.
Therefore, for all $\varepsilon>0$, if we take $N=\frac{\sqrt{2}}{\varepsilon}$, then for all $n>N$, $|x_n|<\varepsilon$.

(b) Define $r_n = 1.\underbrace{41421\ldots}_{\text{first } n \text{ digits of decimal approximation of } \sqrt{2}}$.

Or more precisely, we define r_n by $r_n = \lfloor \sqrt{2}\cdot 10^n \rfloor / 10^n$, where $\lfloor a \rfloor$ represents the largest integer less than or equal to a. Then $r_n \in \mathbb{Q}$.

Claim: $\lim_{n\to\infty} r_n = \sqrt{2}$. Note that
$|r_n - \sqrt{2}| = 10^{-n} |\lfloor \sqrt{2}\cdot 10^n \rfloor - \sqrt{2}\cdot 10^n| \le 10^{-n}$,
and $10^{-n} < \varepsilon \iff \frac{1}{\varepsilon} < 10^n \iff \log_{10}\frac{1}{\varepsilon} < n$.
Therefore, for all $\varepsilon>0$, if we take $N = \log_{10}\frac{1}{\varepsilon}$, then for all $n>N$, $|r_n - \sqrt{2}| < \varepsilon$.

12)

a) 0

b) 2

c) 0

d) 2

e) 0+0 = 0

f) 0+2 = 2

g) 2+2 = 0

h) 2

i) 1

j) 3

k) 0