Math 117: Homework 7

Due Sunday, March 3rd at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1

Suppose $t \in \mathbb{R} \cup \{\pm\infty\}$. Prove that t is a subsequential limit of s_n if and only if $-t$ is a subsequential limit of $-s_n$.

Question 2*

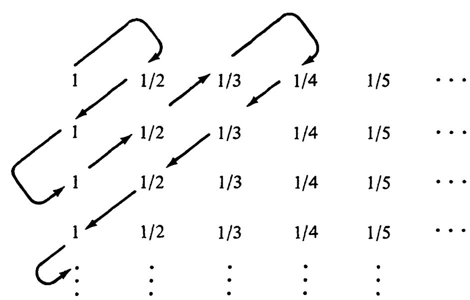
Prove the following generalized squeeze lemma:

LEMMA 1 (Generalized Squeeze). *Suppose* $a_n \leq b_n \leq c_n$ *for all but finitely many* n. *If* $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n$, *then* $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = \lim_{n\to+\infty} c_n$.

Note that the lemma does not assume that the sequences a_n and c_n converge, only that their limits exist and are equal.

Question 3*

Let s_n be the sequence defined in the following figure from the textbook:



(a) Find the set S of subsequential limits of s_n. Justify your answer.

(b) Determine $\limsup s_n$ and $\liminf s_n$. Justify your answer.

Question 4*

Suppose s_n is a bounded sequence and define $s = \sup\{s_n : n \in \mathbb{N}\}$.

(a) State the definition of a bounded sequence.

(b) Suppose $s_n < s$ for all $n \in \mathbb{N}$. Prove that, for all $k \in \mathbb{N}$, the set $B_k := \{s_n : s_n > s - \frac{1}{k}\}$ has infinitely many elements.

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(c) Suppose $s_n < s$ for all $n \in \mathbb{N}$. Use part (b) to show that there exists a subsequence of s_n converging to s.

(d) Now suppose $s_n = s$ for some $n \in \mathbb{N}$. Give an example of such a sequence that doesn't have a subsequence converging to s.

Question 5

(a) Suppose s_n has a subsequence s_{n_k} that is bounded. Show that this implies s_n has a convergent subsequence.

(b) Suppose that s_n has no convergent subsequences. Prove that $\lim_{n\to+\infty} |s_n| = +\infty$.

(Hint: prove the result by contradiction, by showing that if $\lim_{n\to+\infty} |s_n| \neq +\infty$, then s_n has a bounded subsequence.)

Question 6*

In this problem, we will consider sequences s_n satisfying the following property:

$$\exists s \in \mathbb{R} \text{ s.t. every subsequence } s_{n_k} \text{ of } s_n \text{ has a further subsequence } s_{n_{k_l}} \text{ satisfying } \lim_{l\to+\infty} s_{n_{k_l}} = s. \tag{*}$$

(a) Prove that if $\lim s_n = s$, then property (*) holds.

(b) Prove that if property (*) holds, then $\lim s_n = s$. (Hint: Use HW6, Q2, part c)

Question 7

Consider two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ with

$$0 \leq a_k \leq b_k \text{ for all } k \in \mathbb{N}.$$

If $\sum_{k=1}^{\infty} a_k = +\infty$, prove that $\sum_{k=1}^{\infty} b_k = +\infty$.

Question 8*

Suppose $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ for $A, B \in \mathbb{R}$.

(a) Use the limit theorems for sequences to prove that $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$.

(b) Use the limit theorems for sequences to prove that for $c \in \mathbb{R}$, $\sum_{k=1}^{\infty} ca_k = cA$.

Question 9* (absolute value of a series)

In general, the expression $\sum_{k=1}^{\infty} a_k$ doesn't always have meaning, since the limit of the corresponding sequence $s_n = \sum_{k=1}^{n} a_k$ doesn't always exist. On the other hand, in this problem you will show that the expression $\sum_{k=1}^{\infty} |a_k|$ always has meaning.

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(a) Prove that $\sum_{k=1}^{\infty} |a_k|$ is either convergent or diverges to $+\infty$.

(Hint: Show that the corresponding sequence $s_n = \sum_{k=1}^{n} |a_k|$ is monotone.)

(b) Prove that if $\sum_{k=1}^{\infty} |a_k|$ is convergent, then $\sum_{k=1}^{\infty} a_k$ is also convergent.

(Hint: By HW2, Q3 you know that $|\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k|$. Combine this fact with the Cauchy Criterion.)

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