Homework 7 Solutions

① t is a subsequential limit of s_n

$\Updownarrow$

there exists a subsequence s_{n_k} of s_n s.t. $\lim_{k\to\infty} s_{n_k} = t$

$\Updownarrow$ ← If t is a real number, this follows since limit of product is product of limit. If $t = \pm\infty$, this follows from result from class.

there exists a subsequence s_{n_k} of s_n s.t. $\lim_{k\to\infty} -s_{n_k} = -t$

$\Updownarrow$

there exists a subsequence t_{n_k} of $-s_n$ s.t. $\lim_{k\to\infty} t_{n_k} = t$

$\Updownarrow$

t is a subsequential limit of s_n

② First, suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = s \in \mathbb{R}$.
Fix $\varepsilon > 0$. There exists N_a, N_c s.t. $n > N_a$ ensures $|a_n - s| < \varepsilon$ and $n > N_c$ ensures $|c_n - s| < \varepsilon$. Furthermore, there exists N s.t. $n > N$ ensures $a_n \leq b_n \leq c_n$. Let $\tilde{N} = \max\{N_a, N_c, N\}$. Then $n > \tilde{N}$ ensures $s - \varepsilon < a_n \leq b_n \leq c_n < s + \varepsilon$, so $|b_n - s| < \varepsilon$.
This shows $\lim_{n\to\infty} b_n = s$.

Next, suppose $\lim_{n\to\infty} a_n = +\infty$. Fix $M>0$.
There exists N_a s.t. $n > N_a$ ensures $a_n > M$.
There exists N s.t. $n > N$ ensures $a_n \le b_n$.
Let $\tilde{N} = \max\{N_a, N\}$. Then $n > \tilde{N}$ ensures $b_n > M$. This shows $\lim_{n\to\infty} b_n = M$.

Finally, suppose $\lim_{n\to\infty} c_n = -\infty$. Then $-c_n \le -b_n \le -a_n$ for all but finitely many n and $\lim_{n\to\infty} -c_n = +\infty$. By the previous case, $\lim_{n\to\infty} -b_n = +\infty$. Thus, $\lim_{n\to\infty} b_n = -\infty$.

3 a Claim: $S = \{0\} \cup \{\frac{1}{l} : l \in \mathbb{N}\}$

Since $s_{n_k} = \frac{1}{k}$ is a subsequence, $0 \in S$. Since $s_{n_k} = \frac{1}{l}$ is a subsequence for all $l \in \mathbb{N}$, $\frac{1}{l} \in S$.

It remains to show no other real number or $\pm\infty$ belongs to S.

Neither $+\infty$ nor $-\infty$ belong to S, since the sequence is bounded.

Suppose $a \in S$ for some $a \in \mathbb{R}$. By the main subsequences theorem, it suffices to show $\exists\ \varepsilon_0 > 0$ so that $|a - s_n| \geq \varepsilon_0$ for all n.

$\boxed{\text{If } a > 1}$, then $|a - s_n| \geq |a - 1| =: \varepsilon_0\ \forall n$

$\boxed{\text{If } a < 0}$, then $|a - s_n| > |a| =: \varepsilon_0\ \forall n$.

$\boxed{\text{If } \frac{1}{\ell} > a > \frac{1}{\ell+1} \text{ for some } \ell \in \mathbb{N}}$ then $|a - s_n| \geq \min\{|a - \frac{1}{\ell}|, |a - \frac{1}{\ell+1}|\} =: \varepsilon_0\ \forall n$

This completes the proof.

(b) $\limsup s_n = \max(S) = 1$

$\liminf s_n = \min(S) = 0$

④ (a) s_n is a bounded sequence if $\exists M > 0$ s.t. $|s_n| < M \ \forall n \in \mathbb{N}$.

(b) Assume for the sake of contradiction that $\exists k \in \mathbb{N}$ s.t. $B_k := \{s_n : s_n > s - \frac{1}{k}\}$ has finitely many elements.

Case 1: B_k has zero elements
Then $s_n \leq s - \frac{1}{k}$ for all $n \in \mathbb{N}$.
This contradicts the fact that s is the least upper bound.

Case 2: B_k has a nonzero number of elements. Then B_k has a maximum $M_k := \max B_k$. Since $s_n < s$ for all n, $M_k < s$. Also, note that if $s_n \notin B_k$, then $s_n \leq s - \frac{1}{k} \leq M_k$. Thus M_k is an upper bound for $\{s_n : n \in \mathbb{N}\}$. Since $M_k < s$, this contradicts that s was the least upper bound.

Therefore, B_k has infinitely many elements for all $k \in \mathbb{N}$.

(c) Fix $\varepsilon > 0$. Choose $k \in \mathbb{N}$ so that $\frac{1}{k} < \varepsilon$.
Then $\{n : |s_n - s| < \varepsilon\} \supseteq \{n : |s_n - s| < \frac{1}{k}\}$

$$= \{n : s - \tfrac{1}{k} < s_n < s + \tfrac{1}{k}\}$$

since $s_n < s \ \forall n$ $= \{n : s - \frac{1}{k} < s_n\}$

Furthermore $|\{n : s - \frac{1}{k} < s_n\}| \geq |B_k|$, since each element s_n in B_k corresponds to at least one index n in $\{n : s - \frac{1}{k} < s_n\}$.

By part (b), we obtain $|\{n : |s_n - s| < \varepsilon\}| = +\infty$. Thus, by the main subsequences theorem, there is a subsequence of s_n converging to s.

(d) Define $s_n = \frac{1}{n}$. Then $s = \sup\{s_n : n \in \mathbb{N}\} = 1$, but since $\lim_{n \to \infty} s_n = 0$, all subsequences of s_n converge to 0.

⑤ ⓐ If s_{n_k} is bounded, by Bolzano-Weierstrass, s_{n_k} must have a convergent subsequence $s_{n_{k_\ell}}$. Since $s_{n_{k_\ell}}$ is also a subsequence of s_n, s_n has a convergent subsequence.

ⓑ Suppose $|s_n|$ does not diverge to $+\infty$. Then $\exists\, M>0$ s.t. $\forall N$, $\exists\, n>N$ for which $|s_n| \le M$. Since $|s_n| \ge 0$ for all $n \in \mathbb{N}$, this implies there exist infinitely many $n \in \mathbb{N}$ for which $0 \le |s_n| \le M$. Consequently, there exists a subsequence s_{n_k} for which $0 \le |s_{n_k}| \le M$ $\forall\, k \in \mathbb{N}$. Therefore s_{n_k} is a bounded sequence, so by part ⓐ, s_n must have a convergent subsequence.

⑥ ⓐ If $\lim s_n = s$, then all subsequences of s_n also converge to s. Hence every subsequence s_{n_k} has a further subsequence $s_{n_{k_\ell}} = s_{n_k}$ that converges to s.

(b) Suppose $\lim s_n \neq s$. Then,
$\exists\, \varepsilon > 0$ s.t. $\forall N$, $\exists\, n > N$ s.t. $|s_n - s| \geq \varepsilon$

First, taking $N = 1$, we have $\exists\, n_1 > 1$ s.t. $|s_{n_1} - s| \geq \varepsilon$. Suppose we have chosen n_{k-1}. Taking $N = n_{k-1}$, we see that $\exists\, n_k > n_{k-1}$ s.t. $|s_{n_k} - s| \geq \varepsilon$.

Therefore there exists a subsequence s_{n_k} s.t. $|s_{n_k} - s| \geq \varepsilon$ $\forall k$. Since s_{n_k} is always at least distance ε from s, no further subsequence of s_{n_k} can converge to s.

(7) If $\sum_{k=1}^{\infty} a_k = +\infty$, then $s_n := \sum_{k=1}^{n} a_k$ diverges to $+\infty$.

Since $0 \leq a_k \leq b_k$, $t_n := \sum_{k=1}^{n} b_k \geq s_n$. The result then follows from the generalized squeeze lemma.

(8)

Define $s_n = \sum_{k=1}^{n} a_k$, $t_n = \sum_{k=1}^{n} b_k$.

Note that our hypotheses ensure s_n converges to $A \in \mathbb{R}$ and t_n converges to $B \in \mathbb{R}$.

(a) $\sum_{k=1}^{\infty} (a_k + b_k) = \lim_{n\to\infty} (s_n + t_n) \overset{\text{limit of sum is sum of limit}}{=} \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n$
$= A + B$

(b) $\sum_{k=1}^{\infty} c a_k = \lim_{n\to\infty} (c s_n) = c \lim_{n\to\infty} s_n = cA.$

(9)

(a) Since $|a_k| \geq 0 \ \forall k \in \mathbb{N}$,
$s_{n+1} = s_n + |a_{n+1}| \geq s_n$. Thus s_n is monotone, so it must either converge or diverge to $+\infty$.
Therefore $\sum_{k=1}^{\infty} |a_k|$ either converges or diverges to $+\infty$.

(b) Suppose $\sum_{k=1}^{\infty} |a_k|$ is convergent, so it satisfies the Cauchy criterion. We will show $\sum_{k=1}^{\infty} a_k$ is convergent by showing it satisfies the Cauchy criterion.

Fix $\varepsilon > 0$. $\exists$ N s.t. $n > m > N$ implies

$$\left|\sum_{k=m+1}^{n} |a_k|\right| < \varepsilon.$$

Since $\left|\sum_{k=m+1}^{n} a_k\right| \le \left|\sum_{k=m+1}^{n} |a_k|\right|$, we have

that $n > m > N$ implies

$$\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon.$$

Thus, $\sum_{k=1}^{\infty} a_k$ satisfies the Cauchy criterion.