Math 117: Homework 8

Due Sunday, March 17th at 11:59pm

Questions followed by * are to be turned in. Questions without * are extra practice. At least one extra practice question will appear on each exam.

Question 1*

Let $f(x) = \sqrt{2+2x}$ and $g(x) = e^x$.

(a) Give the domains of $f+g$, fg, $f \circ g$, and $g \circ f$.

(b) Are the functions $f \circ g$ and $g \circ f$ equal?

(c) Are the expressions $f \circ g(-2)$ and $g \circ f(-2)$ meaningful?

Question 2*

(a) Prove that, for any $c \in \mathbb{R}$, the constant function $f(x) = c$ is continuous. Prove that, for any $k \in \mathbb{R}$, the function $g(x) = kx$ is continuous.

(b) Consider the function

$$f(x) = \begin{cases} 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Prove that $f(x)$ is not continuous.

Question 3

Prove that the following functions are discontinuous at $x_0 = 0$:

(a) $f(x) = \begin{cases} x+1 & \text{if } x > 0 \\ x-1 & \text{if } x \leq 0 \end{cases}$

(b) $g(x) = \begin{cases} \cos\left(\frac{\pi}{x}\right) & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$

(c) $\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ (This function is known as the *signum function*.)

Question 4*

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Prove that f is continuous at x_0 in $\operatorname{dom}(f)$ if and only if for every sequence x_n in $\operatorname{dom}(f) \setminus \{x_0\}$ that converges to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(Hint: To show that if f satisfies the above criteria then it is continuous at x_0, proceed by contradiction. Using HW6, Q2(c), explain why there exists $\epsilon > 0$ and a subsequence x_{n_k} so that $|f(x_{n_k}) - f(x_0)| \geq \epsilon \; \forall k \in \mathbb{N}$, while $\lim_{k\to+\infty} x_{n_k} = x_0$. Explain why this is a contradiction.)

Question 5

Let f be a real-valued function with $\text{dom}(f) \subseteq \mathbb{R}$. Prove that the following are equivalent:

(i) f is continuous at x_0

(ii) for every *monotonic* sequence x_n in $\text{dom}(f)$ that converges to x_0, we have $\lim f(x_n) = f(x_0)$.

(Hint: To prove that (ii) implies (i), proceed by contradiction and use HW6, Q2(c).)

Question 6*

You may assume that the following functions are continuous on their domains: $\sin(x), \cos(x), e^x, 2^x, \log(x)$ for $x > 0$, and x^p for $x > 0$, where p is any real number. (We use $\log(x)$ to denote the natural logarithm.) You may also assume that the constant function $f(x) = c$ is continuous for any $c \in \mathbb{R}$.

For the following functions, state the domain of each function and prove that the function is continuous on its domain.

(a) $f(x) = \cos(1 - (\log(x))^2)$

(b) $g(x) = \frac{1}{x^2} \cos\left(\frac{1}{1-x}\right)$ for $x \neq 0, 1$

Question 7

(a) Prove that $f(x) = |x|$ is a continuous function on $\mathbb{R}$. (**Hint:** use the *reverse triangle inquality.*)

(b) Use part (b) and the theorem about composition of continuous functions to prove that if g is continuous at $x_0 \in \text{dom}(g)$, then $|g|$ is continuous at x_0.

Question 8*

Prove that $xe^x = 2$ for some $x \in (0, 1)$. You may assume without proof that the functions $f(x) = x$ and $g(x) = e^x$ are continuous.

Note: this problem is an application of the intermediate value theorem, which you will learn about in Thursday's lecture.