Homework 7 Solutions

1) a) $\mathrm{dom}(f+g) = [1,+\infty)$, $\mathrm{dom}(fg) = [1,+\infty)$,
$\mathrm{dom}(f\circ g) = \mathbb{R}$, $\mathrm{dom}(g\circ f) = [-1,+\infty)$

b) No. $f\circ g(x) = \sqrt{2+2e^x} \neq e^{\sqrt{2+2x}} = g\circ f(x)$

c) $f\circ g(-2) = \sqrt{2+2/e^2}$
$g\circ f(-2)$ is not meaningful since $-2 \notin \mathrm{dom}(f)$

2) a) Fix $x_0 \in \mathbb{R}$ and a sequence x_n converging to x_0. We must show $f(x_n)$ converges to $f(x_0)$. Since $f(x_n) = c$ for all $n \in \mathbb{N}$ and $f(x_0) = c$, this is trivially true.

b) Take $x_0 = 0$ and $x_n = \frac{1}{n}$. Then $\lim_{n\to\infty} x_n = x_0$ but $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} n = +\infty$ $\neq 0 = f(x_0)$.

③ Define $x_n = \frac{1}{n}$, so x_n lies in the domain of all three functions and $\lim_{n\to\infty} x_n = x_0$.

ⓐ $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} \frac{1}{n} + 1 = 1 \neq -1 = f(x_0)$, so f is not continuous.

ⓑ $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(\frac{1}{n}) = \lim_{n\to\infty} \cos(\pi n) = \lim_{n\to\infty} (-1)^n$ doesn't exist, so it does not equal $f(x_0)=1$. Hence, f is not continuous.

ⓒ $\lim_{n\to\infty} sgn(x_n) = \lim_{n\to\infty} sgn(\frac{1}{n}) = \lim_{n\to\infty} 1 \neq 0 = sgn(0)$. Hence sgn is not continuous.

④ Suppose f is cts at x_0. By defn of cty, for every sequence $x_n \in dom(f)$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Since $(dom(f) \setminus \{x_0\}) \subseteq dom(f)$, for every sequence $x_n \in dom(f)\setminus\{x_0\}$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

(*) Now, suppose that for every sequence $x_n \in \mathrm{dom}(f)\setminus\{x_0\}$ s.t. $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = f(x_0)$. Let y_n be an arbitrary sequence in $\mathrm{dom}(f)$ s.t. $\lim_{n\to\infty} y_n = x_0$.

We must show $\lim_{n\to\infty} f(y_n) = f(x_0)$. Assume, for the sake of contradiction, that $f(y_n)$ does not converge to $f(x_0)$.

By HW6, Q2(c), $\exists\, \varepsilon > 0$ and a subsequence $f(y_{n_k})$ such that $|f(y_{n_k}) - f(x_0)| \geq \varepsilon \ \forall\, k \in \mathbb{N}$. (**) Note that this is only possible if $y_{n_k} \neq x_0$ for all $k \in \mathbb{N}$. However this means $y_{n_k} \in \mathrm{dom}(f)\setminus\{x_0\}$, and since y_{n_k} is a subsequence of the convergent sequence y_n, $\lim_{k\to\infty} y_{n_k} = x_0$

By assumption (*), this implies $\lim_{k\to\infty} f(y_{n_k}) = f(x_0)$. This contradicts (**).

(5) (i) $\Rightarrow$ (ii)

If f is continuous at x_0, then for <u>all</u> sequences x_n in $\mathrm{dom}(f)$ that converge to x_0, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$, so in particular this also holds for monotonic sequences.

$\neg$(i) $\Rightarrow$ $\neg$(ii)

Suppose x_n is a sequence in $\mathrm{dom}(f)$ that converges to x_0 for which $\lim_{n\to\infty} f(x_n) \neq f(x_0)$. By Homework 6, Q2 (c), there exists $\varepsilon > 0$ and a subsequence x_{n_k} so that $|f(x_{n_k}) - f(x_0)| \geq \varepsilon$ for all k. Since $\lim_{n\to\infty} x_n = x_0$, we also have $\lim_{k\to\infty} x_{n_k} = x_0$. Because any sequence has a monotone subsequence, there is a further subsequence $x_{n_{k_\ell}}$ that is monotone and satisfies $\lim_{\ell\to\infty} x_{n_{k_\ell}} = x_0$. However, $|f(x_{n_{k_\ell}}) - f(x_0)| \geq \varepsilon$ for all ℓ. Hence $\lim_{\ell\to\infty} f(x_{n_{k_\ell}}) \neq f(x_0)$, so (ii) fails.

6 (a) $\mathrm{Dom}(f) = (0, +\infty)$.

Let $f_1(x) = \log(x)$, $f_2(x) = x^2$, $f_3(x) = -1$, $f_4(x) = 1$, and $f_5(x) = \cos(x)$. All of these are continuous on their domains. Hence, so are the following functions...

$$f_2 \circ f_1(x) = \log(x)^2$$
$$(f_3(x))(f_2 \circ f_1(x)) = -\log(x)^2$$
$$f_4(x) + (f_3(x))(f_2 \circ f_1(x)) = 1 - \log(x)^2$$
$$f_5(f_4(x) + (f_3(x))(f_2 \circ f_1(x))) = f(x)$$

(b) $\mathrm{Dom}(g) = \mathbb{R} \setminus \{0, 1\}$

Let $f_1(x) = x$, $f_2(x) = -1$, $f_3(x) = 1$, $f_4(x) = \cos(x)$, $f_5(x) = \frac{1}{x^2}$. All of these functions are continuous on their domains. Hence, so are the following functions

$$(f_2(x))(f_1(x)) = -x$$
$$f_3(x) + (f_2(x))(f_1(x)) = 1 - x$$
$$f_3(x) / f_3(x) + (f_2(x))(f_1(x)) = \frac{1}{1-x}$$
$$f_4\left(\frac{f_3(x)}{f_3(x) + f_2(x) f_1(x)}\right) = \cos\left(\frac{1}{1-x}\right)$$
$$f_5(x)\, f_4\left(\frac{f_3(x)}{f_3(x) + f_2(x) f_1(x)}\right) = g(x)$$

(7)

(a) Fix $x_0 \in \mathbb{R}$. Let x_n be a sequence converging to x_0. Then, by the Reverse Triangle Inequality,

$$||x_n| - |x_0|| \leq x_n - x_0$$

Fix $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x_0$, $\exists$ N s.t. $n > N$ ensures $|x_n - x_0| < \varepsilon$. Thus $n > N$ ensures $||x_n| - |x_0|| < \varepsilon$. This shows $\lim_{n\to\infty} |x_n| = |x_0|$.

Since x_n and x_0 were arbitrary, this shows f is continuous.

(b) By Thm, the composition of continuous functions is continuous. Since $f(x) = |x|$ is continuous on all of $\mathbb{R}$, if g is a continuous function at $x_0 \in \text{dom}(g)$, then $f \circ g = |g|$ is continuous at x_0.

(8) Since the product of continuous functions is continuous, $h(x) = xe^x$ is continuous. Since $h(0)=0$ and $h(1)=e>2$, the IVT ensures $\exists\ x\in(0,1)$ so that $h(x)=2$.