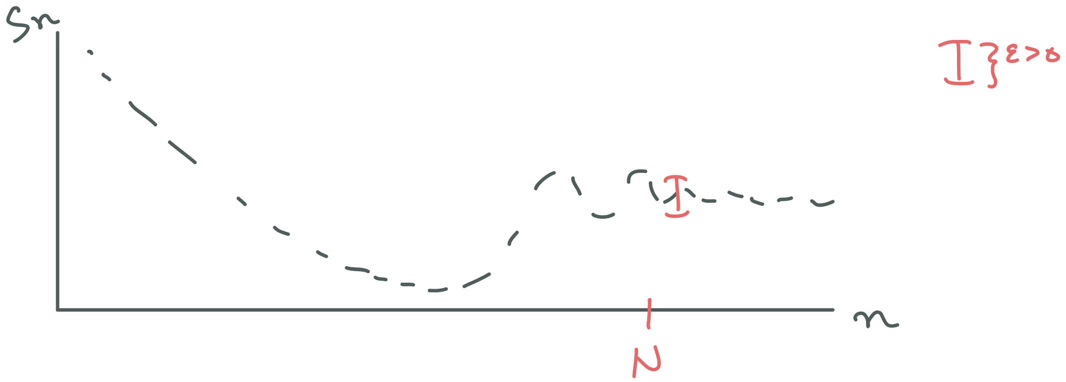
Lecture 10

Def (Cauchy sequence): A sequence s_n is a Cauchy sequence if for all $\varepsilon>0$, there exists $N\in\mathbb{R}$ s.t. $m,n>N$ ensures $|s_n-s_m|<\varepsilon$



How do Cauchy sequences fit in with the types of sequences we already know?

Lemma: Convergent sequences are Cauchy sequences

Pf: Suppose s_n is a convergent sequence, that is $\lim_{n\to\infty} s_n = s$, for $s\in\mathbb{R}$. Fix $\varepsilon>0$. Since $\lim_{n\to\infty} s_n = s$, $\exists N$ s.t. $n>N$ ensures $|s_n-s|<\frac{\varepsilon}{2}$. Thus, for $m,n>N$, we have

$$|s_n-s_m| = \underbrace{|s_n-s+s-s_m|}_{\text{add and subtract}} \underset{\Delta\text{ ineq}}{\leq} |s_n-s|+|s_m-s| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, s_n is Cauchy. $\square$

Lemma: Cauchy sequences are bounded.

The proof is similar to the proof that convergent sequences are bounded.

Remark: Recall the reverse triangle inequality: for $a, b \in \mathbb{R}$,

$$||a| - |b|| \leq |a-b|$$

Since $x \leq |x|$ $\forall x \in \mathbb{R}$, we have $|a| - |b| \leq |a-b|$.

Pf: Let $\varepsilon = 2$. Since s_n is Cauchy, $\exists$ N s.t. $m, n > N$ ensures $|s_n - s_m| < 2$, which implies by reverse triangle inequality $|s_n| - |s_m| < 2 \Leftrightarrow |s_n| < |s_m| + 2$.

In particular, if $n > N$, we have $\lceil N \rceil + 1 > N$, so $|s_n| < 2 + |s_{\lceil N \rceil + 1}|$. Define

$$M = \max\{2 + |s_{\lceil N \rceil + 1}|, |s_1|, |s_2|, \ldots, |s_{\lceil N \rceil}|\}.$$

Then we have $|s_n| < M$ for all $n \in \mathbb{N}$. Thus, s_n is a bounded sequence. $\square$

MAJOR THEOREM #4

Thm: A sequence is convergent iff it is Cauchy.

Here you must know what the limit is and show elts of sequence get close to it.

Here you must know that elts of sequence "bunch up". (Don't need to know what they are bunching up around)

Remark:

- If $s_n \le b$ for all but finitely many n and the limit of s_n exists, then $\lim_{n\to\infty} s_n \le b$.
- If $a \le b + \varepsilon$ for all $\varepsilon > 0$, then $a \le b$.

Pf:

- We already proved that convergent sequences are Cauchy sequences, so it remains to show that Cauchy sequences converge.
- Suppose s_n is Cauchy. By theorem from last time, it suffices to show $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n$ to conclude that $\lim_{n\to\infty} s_n$ exists. Since we already showed Cauchy sequences are bounded,

it would be impossible for $\lim_{n\to\infty} s_n$ to equal $+\infty$ or $-\infty$. Thus, the sequence must converge.

- Fix $\varepsilon > 0$. Since s_n is Cauchy, $\exists\, N$ s.t. $n, m > N$ ensures $|s_n - s_m| < \varepsilon$ $\iff s_m - \varepsilon < s_n < s_m + \varepsilon$.

Thus, for $m > N$, we have
$a_N = \sup\{s_n : n > N\} \le s_m + \varepsilon \iff a_N - \varepsilon \le s_m$.

Thus, for $m > N$, we have
$a_N - \varepsilon \le \inf\{s_m : m > N\} = b_N$.

Since a_N is a decreasing sequence and b_N is an increasing sequence, for all $k > N$,

$$a_k - \varepsilon \le a_N - \varepsilon \le b_N \le b_k.$$

By Remark,

$$\limsup_{n\to\infty} s_n - \varepsilon = \lim_{k\to\infty} a_k - \varepsilon \le b_N \le \lim_{k\to\infty} b_k = \liminf_{n\to\infty} s_n.$$

Thus, $\limsup_{n\to\infty} s_n \le \liminf_{n\to\infty} s_n + \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, by remark, $\limsup_{n\to\infty} s_n \le \liminf_{n\to\infty} s_n$.

- We always have $\liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n$. Thus $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n$. $\square$

Types of Sequences:

	MONOTONE	NOT MONOTONE	
BOUNDED	$s_n = \frac{1}{n}$	$s_n = \frac{(-1)^n}{n}$	$s_n = (-1)^n$
UNBOUNDED	$s_n = n^3$	$s_n = \begin{cases} (-1)^n & n \le 4 \\ n & n \ge 5 \end{cases}$	$s_n = (-1)^n n$

CAUCHY SEQUENCES
$\Updownarrow$
CONVERGENT SEQUENCE

DIVERGE TO $+\infty$ OR $-\infty$

THE LIMIT EXISTS