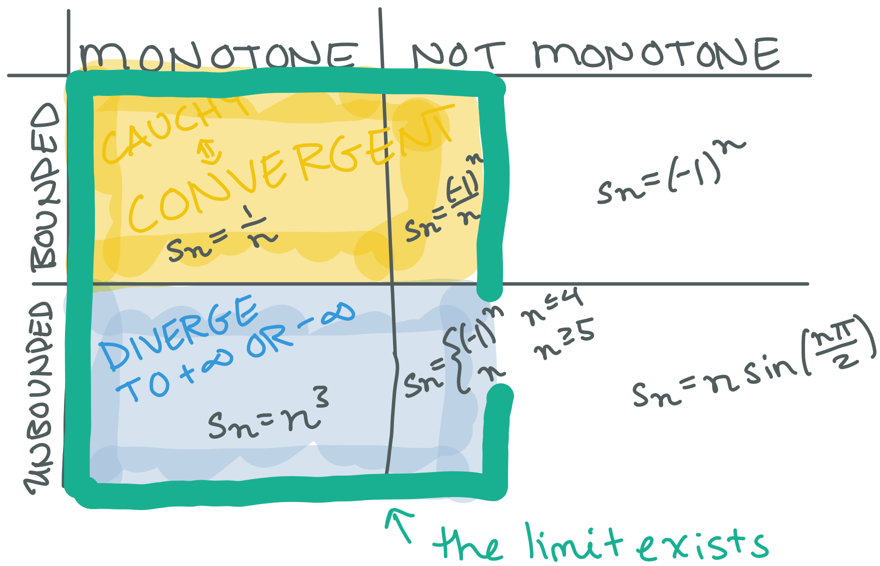
Lecture 11

Types of Sequences



Goal: we know a lot about monotone sequences... what can we say about bounded sequences.

First, recall...

Def (sequence): A sequence is a function whose domain is a set of the form $\{m, m+1, m+2, \dots\}$ for some $m \in \mathbb{Z}$. We study sequences whose range is $\mathbb{R}$.

Remark: While we could write $s(n)$, we use s_n to emphacize that sequences are a special type of functions!

Now, we will define the notion of subsequence..

Def (subsequence): Consider a sequence s_n. For any sequence n_k of natural numbers satisfying $n_1 < n_2 < n_3 < \dots$, a sequence of the form s_{n_k} is a subsequence of s_n.

Remark: We could write s_n as $s(n)$, n_k as $n(k)$, and s_{n_k} as $s(n(k))$.

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

$$\text{Ex①}: s_n = (\overset{n=1}{-1}, \overset{n=2}{2}, \overset{n=3}{-3}, \overset{n=4}{4}, \dots, \overset{n}{(-1)^n n}, \dots)$$

$$s_{n_k} = (\overset{k=1}{-1}, \overset{k=2}{-3}, -5, \dots, \overset{k}{(-1)^{(2k-1)}(2k-1)}, \dots)$$

$$n_k = (\overset{k=1}{1}, \overset{k=2}{3}, \overset{k=3}{5}, \dots, \overset{k}{2k-1}, \dots)$$

Note that

$$a_N = \sup\{s_n : n > N\} = (\overset{N=1}{+\infty}, \overset{N=2}{+\infty}, \dots, \overset{N}{+\infty}, \dots)$$

$$b_N = \inf\{s_n : n > N\} = (-\infty, -\infty, \dots, -\infty, \dots)$$

Ex 2: $s_n = (\overset{n=1}{1}, \overset{n=2}{\tfrac{1}{2}}, \overset{n=3}{3}, \overset{n=4}{\tfrac{1}{4}}, \dots, \overset{n}{n^{(-1)^{n+1}}}, \dots)$

$$s_{n_k} = (\overset{k=1}{\tfrac{1}{2}}, \overset{k=2}{\tfrac{1}{4}}, \overset{k=3}{\tfrac{1}{6}}, \dots, (2k)^{-1}, \dots)$$

$$n_k = (\overset{k=1}{2}, \overset{k=2}{4}, \overset{k=3}{6}, \dots, \overset{k}{2k}, \dots)$$

$$a_N = \sup\{s_n : n > N\} = (\overset{N=1}{+\infty}, \overset{N=2}{+\infty}, \dots)$$

$$b_N = \inf\{s_n : n > N\} = (0, 0, \dots)$$

Limits of Subsequences

Lemma: Given a sequence s_n, $n \in \mathbb{N}$, if s_{n_k} is a subsequence, then $n_k \geq k$ for all $k \in \mathbb{N}$.

Pf: Base case: When $k=1$, $n_1 \geq 1$ since $n_k \in \mathbb{N}$ for all k.

Inductive step: Assume $n_{k-1} \geq k-1$. Since $n_k > n_{k-1}$, we have $n_k \geq n_{k-1}+1 \geq k$. $\square$

Def: (subsequential limit) A subsequential limit of a sequence s_n is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of s_n.

Ex: $s_n = (1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots)$

0 and $+\infty$ are subsequential limits

Thm: If a sequence s_n converges to a limit s, then every subsequence also converges to s.

Pf: Let s_{n_k} be an arbitrary subsequence of s_n. Fix $\varepsilon>0$. Since $\lim_{n\to\infty} s_n = s$, $\exists$ N s.t. $n>N$ ensures $|s_n - s|<\varepsilon$. If $k>N$, then $n_k \geq k > N$, so $|s_{n_k} - s| < \varepsilon$. Since $\varepsilon>0$ was arbitrary, we have $\lim_{k\to\infty} s_{n_k} = s$.

Ex: $s_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

$\{0\}$ is the set of <u>all</u> subsequential limits

<u>Thm</u> (main subsequences theorem)

Let s_n be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$

[The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon>0$]

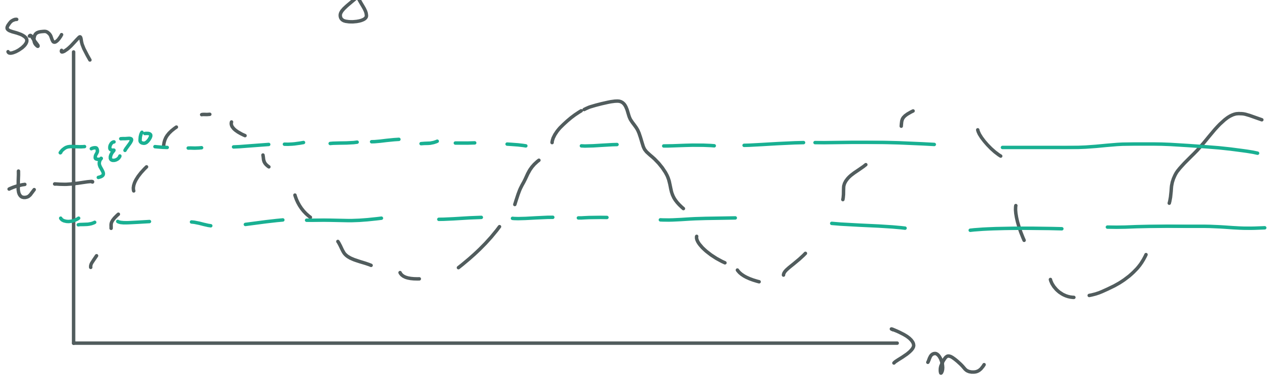
if and only if

[t is a subsequential limit of s_n.]

(b) s_n is unbounded above $\iff$ $+\infty$ is a subseq. limit.

(c) s_n is unbounded below $\iff$ $-\infty$ is a subseq. limit.

Mental image (a):



Mental image (b) + (c)

<u>Lemma</u>: If s_n is unbounded above, the set $\{n : s_n > M\}$ is infinite for all $M > 0$.

Pf: Assume, for the sake of contradiction, that there exists $M > 0$ for which $\{n : s_n > M\}$ is finite. Define

$$s_{max} = \max\{s_n : s_n > M\}.$$

Then define $\tilde{M} = \max\{s_{max}, M\}$

- if $s_n > M$, $s_n \le s_{max} \le \tilde{M}$
- if $s_n \le M$, $s_n \le \tilde{M}$.

Thus, for all $n \in \mathbb{N}$, $s_n \le \tilde{M}$, so s_n is bounded above, which is a contradiction. $\square$

Pf of Main Subsequences Theorem

(a) Suppose [The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$].

We can construct a subsequence of s_n in the following way:

Choose s_{n_1} so that $|s_{n_1} - t| < 1$.

Choose s_{n_2} so that $|s_{n_2} - t| < \frac{1}{2}$ and $n_2 > n_1$.

$\vdots$

Choose s_{n_k} so that $|s_{n_k} - t| < \frac{1}{k}$ and $n_k > n_{k-1}$.

Note that $|s_{n_k} - t| < \frac{1}{k} \Leftrightarrow t - \frac{1}{k} < s_{n_k} < t + \frac{1}{k}$ for all $k \in \mathbb{N}$. So by the squeeze lemma, $t \le \lim_{k\to\infty} s_{n_k} \le t$, so $\lim_{k\to\infty} s_{n_k} = t$ and t is a subsequential limit.

Now, suppose [t is a subsequential limit of s_n.]

Fix $\varepsilon > 0$. Since there exists a subsequence s_{n_k} that converges to t, there exists N s.t. $k > N$ ensures $|s_{n_k} - t| < \varepsilon$.

Therefore, $\{n_k : k > N\} \subseteq \{n : |s_n - t| < \varepsilon\}$.

Since $\{n_k : k > N\}$ is infinite, so is $\{n : |s_n - t| < \varepsilon\}$.

(b) Suppose [s_n is unbounded above].
By the lemma, for all $M>0$, $\{n : s_n > M\}$ is infinite. Hence, we may construct a subsequence as follows.
Choose n_1 so that $s_{n_1} > 1$.
Choose n_2 so that $s_{n_2} > 2$ and $n_2 > n_1$.
$\vdots$
Choose n_k so that $s_{n_k} > k$ and $n_k > n_{k-1}$.

Fix $\tilde{m} > 0$. For $k > \tilde{m}$, $s_{n_k} > k > \tilde{m}$.
Since $\tilde{m}$ was arbitrary, $\lim_{k\to\infty} s_{n_k} = +\infty$.
Thus $+\infty$ is a subsequential limit.

Suppose [$+\infty$ is a subsequential limit].
Assume, for the sake of contradiction, that s_n is bounded above, that is there exists $M>0$ s.t. $s_n \leq M$ for all $n \in \mathbb{N}$. Take s_{n_k} s.t. $\lim_{k\to\infty} s_{n_k} = +\infty$.
Then $s_{n_k} \leq M$ for all $k \in \mathbb{N}$. This is a contradiction.

(c) Note that
[s_n is unbounded below]
$\Updownarrow$

[$-s_n$ is unbounded above]

$\Updownarrow$ (b)

[$+\infty$ is a subsequential limit of $-s_n$]

$\Updownarrow$

[$-\infty$ is a subsequential limit of s_n] $\square$