Lecture 11

Office hours change -No Wednesday, yes Friday, 9-10am

Def (subsequence): Consider a sequence s_n. For any sequence n_k of natural numbers satisfying $n_1 < n_2 < n_3 < \dots$, a sequence of the form s_{n_k} is a **subsequence** of s_n.

Remark: We could write s_n as $s(n)$, n_k as $n(k)$, and s_{n_k} as $s(n(k))$.

Lemma: Given a sequence s_n, $n \in \mathbb{N}$, if s_{n_k} is a subsequence, then $n_k \geq k$ for all $k \in \mathbb{N}$.

Def: (subsequential limit) A **subsequential limit** of a sequence s_n is any real number or symbol $+\infty$ or $-\infty$ that is the limit of some subsequence of s_n.

Thm: If a sequence s_n converges to a limit s, then every subsequence also converges to s.

Thm (main subsequences theorem)

Let s_n be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$

[The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$]

if and only if

[t is a subsequential limit of s_n.]

(b) s_n is unbounded above $\iff$ $+\infty$ is a subseq. limit.

(c) s_n is unbounded below $\iff$ $-\infty$ is a subseq. limit.

Negation of Thm.

(a) Let $t \in \mathbb{R}$

[For some $\varepsilon > 0$, the set $\{n : |s_n - t| < \varepsilon\}$ is finite.]

if and only if

[t is not a subsequential limit of s_n.]

(b) s_n is bounded above $\iff$ $+\infty$ is not a subseq. limit.

(c) s_n is bounded below $\iff$ $-\infty$ is not a subseq. limit.

Informally, a subsequence is any infinite collection of elements from the original sequence, listed in order.

$$\begin{matrix} & n=1 & n=2 & n=3 & n=4 & & n & \\ \text{Ex ①: } s_n = (& -1, & 2, & -3, & 4, & \ldots, & (-1)^n n, & \ldots) \end{matrix}$$

$$\begin{matrix} & k=1 & k=2 & k=3 & k=4 & & k & \\ n_k = (& 3, & 6, & 9, & 12, & \ldots, & 3k, & \ldots) \end{matrix}$$

$$\begin{matrix} & k=1 & k=2 & k=3 & & & \\ s_{n_k} = (& -3, & 6, & -9, & \ldots, & (-1)^{3k}(3k), & \ldots) \end{matrix}$$

Note that

$$\begin{matrix} & N=1 & N=2 & N=3 & \\ a_N = \sup\{s_n : n > N\} = (& +\infty, & +\infty, & +\infty, & \ldots) \end{matrix}$$

$$b_N = \inf\{s_n : n > N\} = (-\infty, -\infty, -\infty, \ldots)$$

Discuss negation of main subsequences theorem and an example to prove that something ISN'T a sub sequential limit, since this is used a lot on HW6