Lecture 12

Thm (main subsequences theorem)
Let s_n be a sequence of real numbers.

(a) Let $t \in \mathbb{R}$

[The set $\{n : |s_n - t| < \varepsilon\}$ is infinite for all $\varepsilon > 0$]

if and only if

[t is a subsequential limit of s_n.]

(b) s_n is unbounded above $\iff$ $+\infty$ is a subseq. limit.

(c) s_n is unbounded below $\iff$ $-\infty$ is a subseq. limit.

Why are subsequences important?

Even though not all sequences are monotone

Thm: Every sequence s_n has a monotonic subsequence.

Pf: We will say that the n^{th} element of a sequence is <u>dominant</u> if it is greater than every element that follows, that is s_n is <u>dominant</u> if $s_n > s_m$ for all $m > n$.

$\boxed{\text{Case 1}}$: Suppose s_n has infinitely many dominant elements.

Define s_{n_k} to be the subsequence of dominant terms. Then $s_{n_k} > s_{n_{k+1}}$ for all $k \in \mathbb{N}$, so s_{n_k} is decreasing, hence monotone.

$\boxed{\text{Case 2}}$: Suppose s_n has finitely many dominant elements.

- Choose n_1 so that s_{n_1} is beyond all of the dominant elements in the sequence.
- Since s_{n_1} is not dominant, there exists $n_2 > n_1$ so that $s_{n_2} \geq s_{n_1}$.
- Since s_{n_k} is not dominant, there exists $n_{k+1} > n_k$ so that $s_{n_{k+1}} \geq s_{n_k}$.

Thus we have found a subsequence that is increasing, hence monotone. $\square$

↙ MAJOR THEOREM 5

Thm (Bolzano-Weierstrass): Every bounded sequence has a convergent subsequence.

Pf: If s_n is a bounded sequence, the previous theorem ensures there exists a subsequence s_{n_k} that is monotonic (and also bounded). Since all bounded, monotone sequences converge, s_{n_k} is convergent. $\square$