Lecture 13

How do subsequences relate to liminf and limsup?

Downside: in general $\underset{\sup\{s_n : n > N\}}{\overset{||}{a_N}}, \underset{\inf\{s_n : n > N\}}{\overset{||}{b_N}}$ are not subsequences of s_n.

Upside:

Thm: For any sequence s_n, $\limsup s_n$ and $\liminf s_n$ are subsequential limits.

Pf: First, we will show $\limsup_{n\to\infty} s_n$ is a subsequential limit.

$\boxed{\text{CASE 1}}$: Suppose $\limsup_{n\to\infty} s_n = -\infty$. Since $\liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n$, then $\liminf_{n\to\infty} s_n = -\infty$, so

$\lim_{n\to\infty} s_n = -\infty$.

$\boxed{\text{CASE 2}}$: Suppose $\limsup_{n\to\infty} s_n = +\infty$, that is $\lim_{N\to\infty} a_N = +\infty$. Fix arbitrary $M>0$. Then there exists N_0 s.t. $N>N_0$ ensures $a_N > M$. Thus M is not an upper bound of $\{s_n : n>N\}$ when $N>N_0$, so there exists $s_{N_1} > M$. Thus s_n is not bounded above. Hence $+\infty$ is a subsequential limit.

$\boxed{\text{CASE 3}}$: Suppose $\limsup_{n\to\infty} s_n = t$ for $t\in\mathbb{R}$, that is $\lim_{N\to\infty} a_N = t$. Fix arbitrary $\varepsilon>0$. We will show $\{n : t-\varepsilon < s_n < t+\varepsilon\} = \{n : |s_n - t| < \varepsilon\}$ is infinite.

By defn of convergence of a_N to t, $\exists\, N_0$ s.t. $N>N_0$ ensures $|a_N - t| < \varepsilon \Rightarrow$ $\sup\{s_n : n>N\} = a_N < t+\varepsilon$. In particular, for $N = \lceil N_0\rceil + 1$, $\sup\{s_n : n > \lceil N_0\rceil + 1\} < t+\varepsilon$. Thus for all $n > \lceil N_0\rceil+1$, $s_n < t+\varepsilon$.

Suppose, for the sake of contradiction, that $\{n : t-\varepsilon < s_n < t+\varepsilon\}$ is finite.
Since we know $n > \lceil N_0 \rceil + 1$ ensures $s_n < t+\varepsilon$, there must be $N_1 > \lceil N_0 \rceil + 1$ for which $s_n \le t-\varepsilon$ for all $n > N_1$.

Then $a_N = \sup\{s_n : n > N\} \le t-\varepsilon$ for $N > N_1$.
This implies $\lim_{N\to\infty} a_N \le t-\varepsilon$. This contradicts that $\lim_{N\to\infty} a_N = \limsup_{n\to\infty} s_n = t$. Therefore, $\{n : t-\varepsilon < s_n < t+\varepsilon\}$ is infinite. Since $\varepsilon > 0$ was arbitrary, by main subseq. theorem, t is a subsequential limit.

Next, we show $\liminf_{n\to\infty} s_n$ is a subsequential limit.

Fact: $\liminf_{n\to\infty} s_n = -\limsup_{n\to\infty} -s_n$

Thus, by what we've already shown, $-\liminf_{n\to\infty} s_n$ is a subsequential limit of $-s_n$

Fact: t is a subseq. limit of s_n
$\iff -t$ is a subseq. limit of $-s_n$

Thus $\liminf_{n\to\infty} s_n$ is a subseq limit of s_n. $\square$

In fact, $\limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} s_n$ aren't just <u>any</u> subsequential limit: they are the largest and smallest subsequential limit.

Recall: squeeze lemma
Given $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$, if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$.

<u>Thm</u>: Let S denote the set of subsequential limits of s_n, then $\limsup s_n = \max(S)$ and $\liminf s_n = \min(S)$.

Pf: By the previous theorem, we have $\limsup_{n\to\infty} s_n \in S$ and $\liminf_{n\to\infty} s_n \in S$, so it suffices to show that, for all $t \in S$, we have $\liminf_{n\to\infty} s_n \le t \le \limsup_{n\to\infty} s_n$. Suppose $\lim_{k\to\infty} s_{n_k} = t$.

Since $n_k \ge k$, $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$ for any $N \in \mathbb{R}$. Thus

$$b_N = \inf\{s_n : n > N\} \le \inf\{s_{n_k} : k > N\}$$

$$\sup\{s_{n_k} : k > N\} \le \sup\{s_n : n > N\} = a_N$$

Sending $N \to \infty$,

$$\liminf_{n\to\infty} s_n = \lim_{N\to\infty} b_N \le \liminf_{k\to\infty} s_{n_k} = t = \limsup_{k\to\infty} s_{n_k} \le \lim_{N\to\infty} a_N = \limsup_{n\to\infty} s_n$$

$\square$