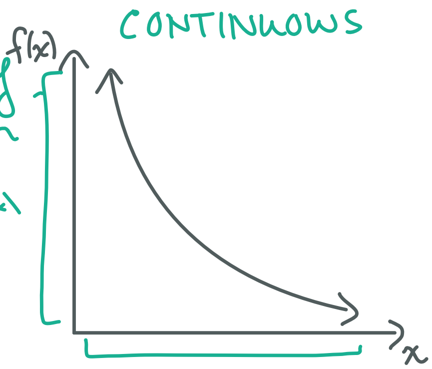
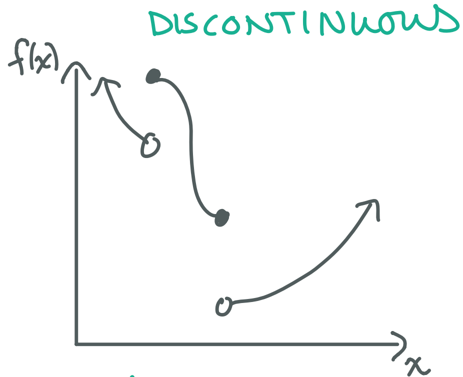
Lecture 14

Now, we will apply the theory of sequences of real numbers to study continuous functions

Heuristically, a function is continuous if it is an "unbroken curve" with "no holes".

range of function set in which f(x) belongs





domain of function is the set of values of x for which $f(x)$ is defined, abbreviate dom(f)

We will study real-valued functions, with $\mathrm{dom}(f) \subseteq \mathbb{R}$

range(f) $\subseteq \mathbb{R}$

Ex: $f(x) = \frac{1}{x}$, $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$ ← we will study this type of fn

$f(x,y) = \left[\frac{x^2+y^2}{5}\right]$, $\mathrm{dom}(f) = \mathbb{R}^2$ ← will study in math 118

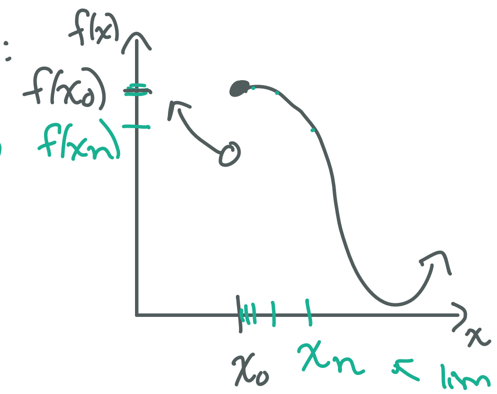
We can make the heuristic notion of continuity precise using sequences.

Def (continuous function):

- A function f is ==continuous at a point x_0== $\in \mathrm{dom}(f)$ if, for every sequence x_n in $\mathrm{dom}(f)$ satisfying $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.
- f is ==continuous on a set S== $\subseteq \mathrm{dom}(f)$ if it is continuous at every point in S.
- f is ==continuous== if it is continuous on all of $\mathrm{dom}(f)$.

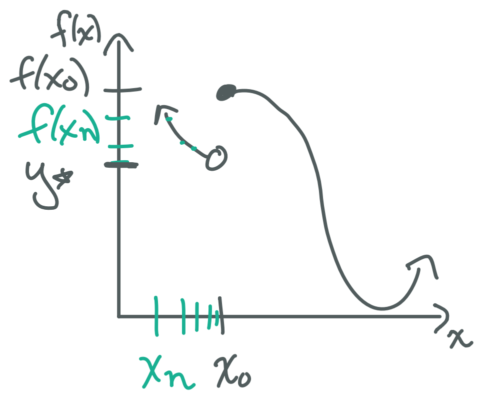
Mental image:

$\lim_{n\to\infty} f(x_n) = f(x_0)$



$\lim_{n\to\infty} x_n = x_0$

$\lim_{n\to\infty} f(x_n) = y_*$
$\neq f(x_0)$



$\lim_{n\to\infty} x_n = x_0$

Remark: If a function $f(x)$ is continuous, you can "pass the limit inside the function":

If x_n and x_0 are in $\mathrm{dom}(f)$ and $\lim_{n\to\infty} x_n = x_0$,

$$\lim_{n\to\infty} f(x_n) = f(x_0) = f\left(\lim_{n\to\infty} x_n\right).$$

Ex: One can show that $f(x) = \sin(x)$ is continuous. Thus, for any convergent sequence x_n,

$$\lim_{n\to\infty} \sin(x_n) = \sin\left(\lim_{n\to\infty} x_n\right).$$

The definition of continuity that appears in most textbooks involves ε's and δ's. The next theorem shows that this is equivalent to our definition.

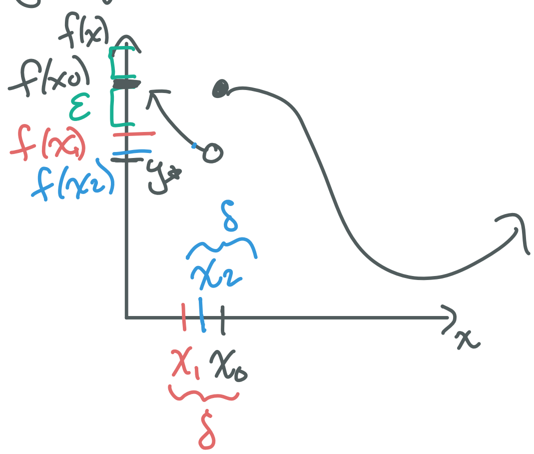
Thm (ε-δ characterization of continuity)
Given f and $x_0 \in \mathrm{dom}(f)$,

I f is continuous at x_0 if and only if

II for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

$\neg$II there exists $\varepsilon > 0$ so that, for all $\delta > 0$, there exists $x \in \mathrm{dom}(f)$ with $|x - x_0| < \delta$ satisfying $|f(x) - f(x_0)| \geq \varepsilon$.

Mental image for ε-δ characterization of cty:



Pf:

Assume (II). Fix x_n in dom(f) s.t. $\lim_{n\to\infty} x_n = x_0$. We aim to show $\lim_{n\to\infty} f(x_n) = f(x_0)$. Fix $\varepsilon > 0$. By (II), there exists $\delta > 0$ so that $|x_n - x_0| < \delta$ implies $|f(x_n) - f(x_0)| < \varepsilon$. Since $\lim_{n\to\infty} x_n = x_0$, there exists N s.t. $n > N$, $|x_n - x_0| < \delta$. Thus, $n > N$ ensures $|f(x_n) - f(x_0)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} f(x_n) = f(x_0)$.

To prove (I) $\Rightarrow$ (II), we will show $\neg$(II) $\Rightarrow \neg$(I). Assume $\neg$(II), that is, there exists $\varepsilon > 0$ so that for all $\delta > 0$, there exists $x \in \text{dom}(f)$ with $|x - x_0| < \delta$ satisfying $|f(x) - f(x_0)| \geq \varepsilon$. In particular, since $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, there exists $x_n \in \text{dom}(f)$ satisfying $|x_n - x_0| < \frac{1}{n}$

and $|f(x_n) - f(x_0)| \geq \varepsilon$. Then $\lim_{n\to\infty} x_n = x_0$, but $\lim_{n\to\infty} f(x_n) \neq f(x_0)$. This shows $\neg$ (I). $\square$

Ex: Consider $f(x) = 3x^2 - 2$, $\mathrm{dom}(f) = \mathbb{R}$

Step 1: prove $f(x)$ is continuous via sequences definition. Fix $x_0 \in \mathbb{R}$. Fix x_n in $\mathbb{R}$ with $\lim_{n\to\infty} x_n = x_0$. Then, by the limit theorems (the limit of product/sum is the product/sum of limits) we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 3(x_n)^2 - 2 = 3(x_0)^2 - 2 = f(x_0).$$

This shows f is continuous at x_0. Since $x_0 \in \mathrm{dom}(f)$ was arbitrary, f is continuous.

- $x_n \to x_0$
- $3x_n \to 3x_0$ (limit of product is product of...)
- $(3x_n)(x_n) \to (3x_0)(x_0)$ (limit of product...)
- $3x_n^2 - 2 \to 3x_0^2 - 2$ (limit of sum...)

Step 2: prove $f(x)$ is continuous via ε-δ characterization of continuity.

Fix $x_0 \in \mathbb{R}$. Fix $\varepsilon > 0$.
Let $\delta = \min\left\{\frac{\varepsilon}{3(1+2|x_0|)}, 1\right\}$, so $\delta \leq \frac{\varepsilon}{3(1+2|x_0|)}$ and $\delta \leq 1$.

Scratchwork:

$$|f(x)-f(x_0)|<\varepsilon \iff |(3x^2-2)-(3x_0^2-2)|<\varepsilon$$
$$\iff 3|x^2-x_0^2|<\varepsilon$$
$$\iff 3|(x-x_0)(x+x_0)|<\varepsilon$$
$$\iff 3|x-x_0||x+x_0|<\varepsilon$$
$$\Leftarrow 3|x-x_0|(1+2|x_0|)<\varepsilon$$
$$\iff |x-x_0|<\underbrace{\frac{\varepsilon}{3(1+2|x_0|)}}_{\delta}$$

$|x|-|x_0|\le|x-x_0|<\delta\le 1$
$\Rightarrow |x|\le 1+|x_0|$
$\Rightarrow |x+x_0|\le|x|+|x_0|\le 1+2|x_0|$

Since $\delta\le 1$, if $|x-x_0|<\delta$, by the reverse triangle inequality, $|x|-|x_0|\le|x-x_0|<\delta\le 1$, so $|x|\le 1+|x_0|$, so $\underbrace{|x+x_0|\le|x|+|x_0|\le 1+2|x_0|.}_{(*)}$

Since $\delta\le\frac{\varepsilon}{3(1+2|x_0|)}$, if $|x-x_0|<\delta$,

$$|x-x_0|<\frac{\varepsilon}{3(1+2|x_0|)} \iff 3|x-x_0|(1+2|x_0|)<\varepsilon$$
$$\overset{(*)}{\Rightarrow} 3|x-x_0||x+x_0|<\varepsilon$$
$$\iff 3|(x-x_0)(x+x_0)|<\varepsilon$$
$$\iff 3|x^2-x_0^2|<\varepsilon$$
$$\iff |(3x^2-2)-(3x_0^2-2)|<\varepsilon$$

$$\iff |f(x) - f(x_0)| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, f is continuous at x_0. Since $x_0 \in \mathrm{dom}(f)$ was arbitrary, f is continuous.