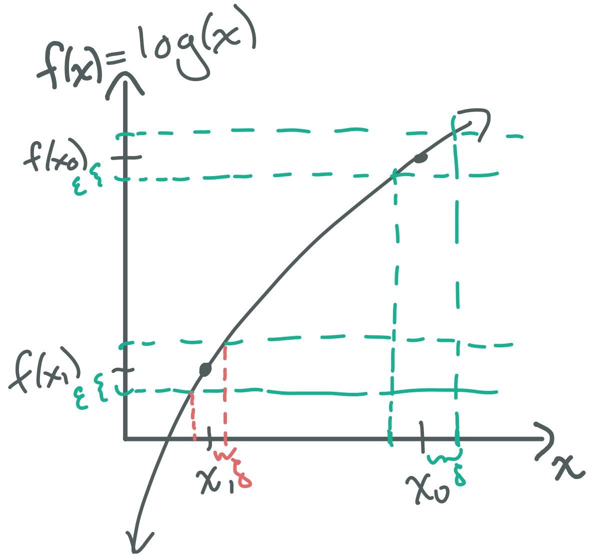
Lecture 14 - Optional

Thm (ε–δ characterization of continuity)
Given f and $x_0 \in \mathrm{dom}(f)$,
$\boxed{f \text{ is continuous at } x_0}$ if and only if
$\boxed{\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that } x \in \mathrm{dom}(f) \text{ and } |x - x_0| < \delta \text{ imply } |f(x) - f(x_0)| < \varepsilon.}$

Remark: The fact that δ often depends on x_0 and ε is an important feature of the ε-δ definition of continuity.

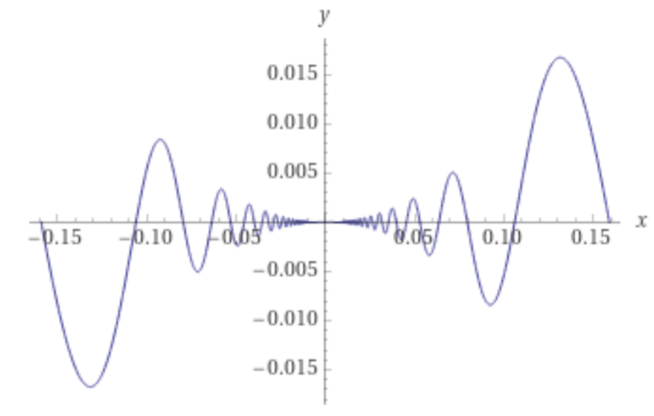


$\mathrm{dom}(f) = (0, +\infty)$

Moral: larger slope at $x_0 \Rightarrow$ smaller δ
smaller slope at $x_0 \Rightarrow$ larger δ

Ex: Consider $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Prove that $f(x)$ is continuous at $x = 0$.



Note that $\text{dom}(f) = \mathbb{R}$.

Pf: (sequences defn of continuity):

Suppose x_n converges to 0. We must show $f(x_n)$ converges to $f(0)$. Fix $\varepsilon > 0$. Note that

$$|f(x_n) - f(0)| = |f(x_n) - 0| = |f(x_n)| \leq |x_n^2 \sin(\tfrac{1}{x_n})|$$

$$= |x_n|^2 |\sin(\tfrac{1}{x_n})| \leq |x_n|^2.$$

Since $x_n \to 0$, we have $|x_n| \to 0$. Since the limit of a product is the product of the limits, $|x_n|^2 \to 0$. Thus $\exists$ N s.t. $n > N$ ensures $||x_n|^2 - 0| = |x_n|^2 < \varepsilon$. Thus $n > N$ ensures $|f(x_n) - f(0)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\lim_{n\to\infty} f(x_n) = f(0)$.

Pf: (ε-δ characterization)
Fix $\varepsilon > 0$. Note that
$|f(x) - f(0)| = |f(x)| \leq |x^2 \sin(\frac{1}{x})| \leq |x|^2 = |x - 0|^2$.
Take $\delta = \sqrt{\varepsilon}$. Then $x \in \mathrm{dom}(f)$ and $|x - 0| < \delta$ ensures $|x - 0|^2 < \delta^2 = \varepsilon$, so $|f(x) - f(0)| < \varepsilon$.