# Lecture 15



Often, the easiest way to prove that a function is continuous is to show that it is a combination of simpler continuous fns

$(f+g)(x) = f(x)+g(x)$, $\operatorname{dom}(f+g) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$
$(fg)(x) = f(x)g(x)$, $\operatorname{dom}(fg) = \operatorname{dom}(f) \cap \operatorname{dom}(g)$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $\operatorname{dom}\left(\frac{f}{g}\right) = \operatorname{dom}(f) \cap \operatorname{dom}(g) \cap \{x : g(x) \neq 0\}$
$(g \circ f)(x) = g(f(x))$, $\operatorname{dom}(g \circ f) = \operatorname{dom}(f) \cap \{x : f(x) \in \operatorname{dom}(g)\}$

Thm (sum, product, quotient cts fns):
If $f$ and $g$ are continuous at $x_0 \in \mathbb{R}$, then
(a) $f+g$ is continuous at $x_0$
(b) $fg$ is continuous at $x_0$
(c) $\frac{f}{g}$ is continuous at $x_0$, provided $g(x_0) \neq 0$.

Pf: Let $h_1 = f+g$, $h_2 = fg$, $h_3 = \frac{f}{g}$. For $i = 1, 2$, and $3$, take $x_n^i \in \mathrm{dom}(h_i)$ that converges to $x_0$. We aim to show $h_i(x_n^i)$ converges to $h_i(x_0)$. Since $f$ and $g$ are continuous functions, $f(x_n^i)$ converges to $f(x_0)$ and $g(x_n^i)$ converges to $g(x_0)$.

- Since the limit of a sum is the sum of the limits, $f(x_n^1) + g(x_n^1) \to f(x_0) + g(x_0)$, that is $h_1(x_n^1) \to h_1(x_0)$. This shows (a).

- Since the limit of a product is the product of the limits, $f(x_n^2)\, g(x_n^2) \to f(x_0)\, g(x_0)$, that is, $h_2(x_n^2) \to h_2(x_0)$. This shows (b).

- Since the limit of a quotient is the quotient of the limits (and we checked that we never divide by zero)

$$\frac{f(x_n^3)}{g(x_n^3)} \longrightarrow \frac{f(x_0)}{g(x_0)},$$

that is $h_3(x_n^3) \to h_3(x_0)$. This shows (c). $\square$

Thm: (composition of cts fns) Suppose $f$ is continuous at $x_0$ and $g$ is continuous at $f(x_0)$. Then $g \circ f$ is continuous at $x_0$.

Pf: Suppose $x_n \in \mathrm{dom}(g \circ f)$ converges to $x_0$. Since $f$ is continuous at $x_0$, $f(x_n) \to f(x_0)$. Since $g$ is continuous at $f(x_0)$, so $g(f(x_n)) \to g(f(x_0))$, that is $g \circ f(x_n) \to g \circ f(x_0)$. $\square$

Going forward, unless otherwise specified... you may use the fact that the following functions are continuous on their domains:

$\sin(x)$, $\cos(x)$, $e^x$, $\log(x)$, $x^p$ for $p \in \mathbb{R}$,
$f(x) = c$ for $c \in \mathbb{R}$

By combining the previous theorems, you may conclude more complicated functions are continuous on their domains:

$e^{-x^2}$, $\sin(4\log(x))$, ...

Just like bounded sequences have important properties, so do bounded functions.

<u>Def</u> (bounded function): $f$ is bounded on $S \subseteq dom(f)$ if there exists $M > 0$ s.t. $|f(x)| \leq M$ for all $x \in S$. We say $f$ is bounded if $f$ is bounded on $dom(f)$.

Remark:

- $s_n$ is a bounded sequence

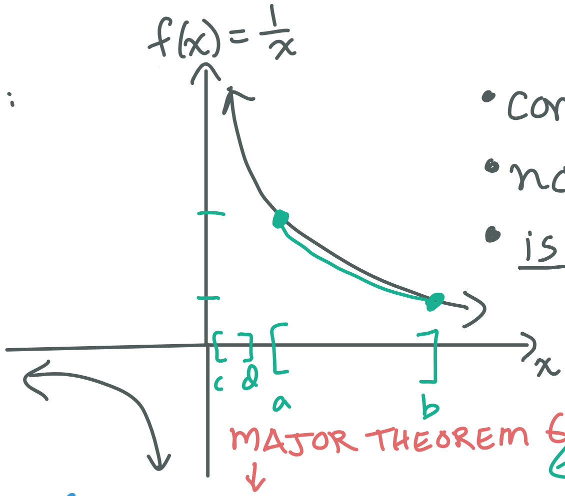
  $\Updownarrow$

  $\{s_n : n \in \mathbb{N}\}$ is a bounded set
- $f$ is a bounded function

  $\Updownarrow$

  $\underbrace{\{f(x) : x \in dom(f)\}}_{= \text{image}(f)}$ is a bounded set

Ex:




- continuous on $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$
- not bounded on $\mathrm{dom}(f)$
- <u>is</u> bounded on any closed interval $[a,b] \subseteq \mathrm{dom}(f)$

this is true for <u>all</u> continuous functions!

MAJOR THEOREM 6

<u>Thm</u> (cts fns attain max and min): A continuous function $f$ on a closed interval $[a,b] \subseteq \mathrm{dom}(f)$ attains its maximum and minimum.

In particular...

(i) its max and min exist (so $f$ is bounded)

(ii) $\exists \; \underset{\text{maximizer}}{x_{max}}, \underset{\text{minimizer}}{x_{min}} \in [a,b]$ so that

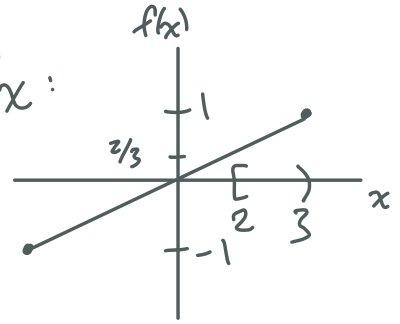
$$\underbrace{f(x_{min})}_{\text{minimum of } f \text{ on } [a,b]} \leq f(x) \leq \underbrace{f(x_{max})}_{\text{maximum of } f \text{ on } [a,b]} \quad \text{for all } x \in [a,b]$$

This theorem makes entire field of optimization possible.

cts function $\qquad\qquad$ closed interval

$$\text{minimize } f(x) \text{ s.t. } x \in [a,b]$$

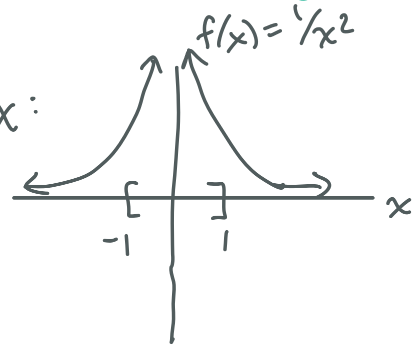
Ex:




- $\{f(x) : x \in [2,3)\} = [\frac{2}{3}, 1)$
- While $f$ does attain its minimum at $x_{min} = 2$, $f(x_{min}) = \frac{2}{3}$, $f$ does not attain its maximum on $[2,3)$.

Moral: cts fns only guaranteed to attain max/min on closed intervals $[a,b]$.

Ex:




- $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$
- $f$ does not attain its max on $[-1,1]$, since $\{f(x) : x \in [-1,1] \setminus \{0\}\} = [1, +\infty)$

Moral: need $[a,b] \subseteq \mathrm{dom}(f)$.

Before we turn to the proof, recall that if $\lim_{n \to \infty} x_n = x_0$ and $x_n \in [a,b]\ \forall n$, then $x_0 \in [a,b]$.

Pf:

$\boxed{\text{Step 1}}$ We will show that $f$ is bounded on $[a,b]$. Assume, for the sake of contradiction, that $f$ is not bounded on $[a,b]$, that is, for all $M>0$, there exists $x \in [a,b]$ s.t. $|f(x)|>M$.

In particular, for all $n \in \mathbb{N}$, there exists $x_n \in [a,b]$ so that $|f(x_n)|>n$. Since $x_n$ is a bounded sequence, by Bolzano-Weierstrass Theorem, it has a subsequence $x_{n_k}$ that converges to $x_0$. Since $x_{n_k} \in [a,b]$ for all $k \in \mathbb{N}$, $x_0 \in [a,b]$. Thus, $\lim_{k\to\infty} x_{n_k} = x_0$, but $|f(x_{n_k})| > n_k \geq k$, so $\lim_{k\to\infty} |f(x_{n_k})| = +\infty$, thus $f(x_{n_k})$ doesn't converge.

Fact: If $\lim_{n\to\infty} t_n = t$, $t \in \mathbb{R}$, then $\lim_{n\to\infty} |t_n| = |t|$.

This contradicts the fact that $f$ is continuous. Therefore $f$ is bounded on $[a,b]$.

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$\boxed{\text{Step 2}}$ We will show that $f$ attains its maximum on $[a,b]$. Since $f$ is bounded $[a,b]$, we know $\{f(x): x\in[a,b]\}$ is a bounded subset of $\mathbb{R}$, so $\sup\{f(x): x\in[a,b]\} = M$ for some $M \in \mathbb{R}$.

Since $M$ is the least upper bound, for all $n\in\mathbb{N}$, $M-\frac{1}{n}$ is not an upper bound, so there exists $x_n \in [a,b]$ s.t. $M \geq f(x_n) > M - \frac{1}{n}$. By the Squeeze Lemma, $\lim_{n\to\infty} f(x_n) = M$.

Since $x_n$ is bounded, by the Bolzano-Weierstrass Theorem, it has a convergent subsequence $x_{n_k}$, with $\lim_{k\to\infty} x_{n_k} = x_0 \in [a,b]$. Since $f$ is continuous, $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$. Since $f(x_{n_k})$ is a subsequence of the convergent sequence $f(x_n)$, $\lim_{k\to\infty} f(x_{n_k}) = f(x_0) = M$.

Therefore $f(x_0) = \sup\{f(x): x\in[a,b]\} \geq f(x)$ for all $x\in[a,b]$, so $x_0$ is the maximizer.

$\boxed{\text{Step 3}}$ We will show that $f$ attains its minimum on $[a,b]$. Since $f$ is continuous and $g(x)=-1$ is continuous, $-f$ is continuous and $\text{dom}(-f)=\text{dom}(f)$, so $[a,b]\subseteq\text{dom}(-f)$. By Step 2, $-f$ will attain its maximum on $[a,b]$,

that is, there exists $x_1\in[a,b]$ so that $-f(x_1)\geq -f(x)$ for all $x\in[a,b]$. Multiplying this inequality by $-1$, we see $f(x_1)\leq f(x)$ for all $x\in[a,b]$. Thus, $f$ attains its minimum at $x_1$