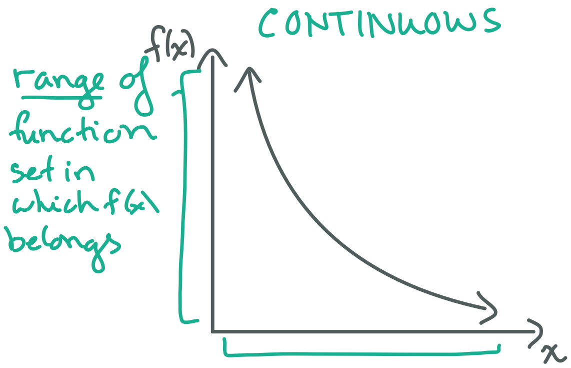
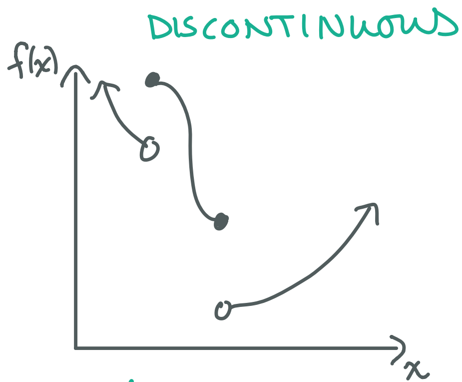
Lecture 15 - Highlights

Review Session: Monday, March 18th, 3:15-5pm, SH 6635





domain of function is the set of values of x for which $f(x)$ is defined, abbreviate dom(f)

Def (continuous function):

- A function f is continuous at a point $x_0 \in \mathrm{dom}(f)$ if, for every sequence x_n in dom(f) satisfying $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.
- f is continuous on a set $S \subseteq \mathrm{dom}(f)$ if it is continuous at every point in S.
- f is continuous if it is continuous on all of dom(f).

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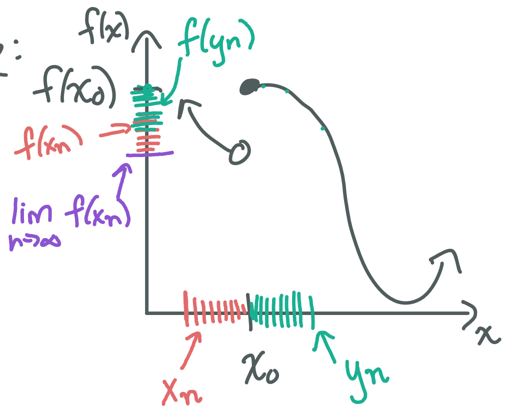
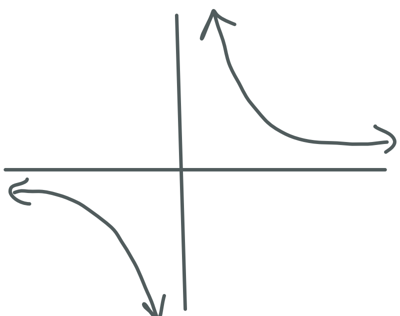
Mental image:



"not continuous" = "discontinuous"

Ex: We will show $f(x)=\frac{1}{x}$ is continuous.

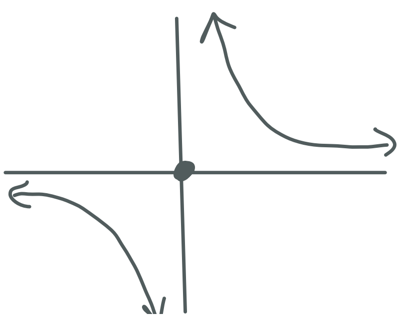
$\mathrm{dom}(f)=\mathbb{R}\setminus\{0\}$



Ex: We will show $f(x)=\begin{cases}\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$ is not cts

$\mathrm{dom}(f)=\mathbb{R}$

It is cts on $S=(0,+\infty)$.



Remark: If a function $f(x)$ is continuous, you can "pass the limit inside the function":

If x_n and x_0 are in $\mathrm{dom}(f)$ and $\lim_{n\to\infty} x_n = x_0$,

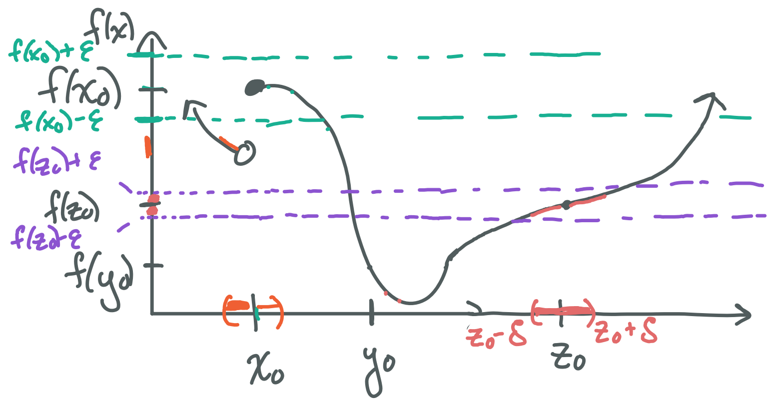
$$\lim_{n\to\infty} f(x_n) = f(x_0) = f\left(\lim_{n\to\infty} x_n\right).$$

<u>Thm</u> (ε–δ characterization of continuity)

Given f and $x_0 \in \mathrm{dom}(f)$,

f is continuous at x_0 if and only if

for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$



Often, the easiest way to prove that a function is continuous is to show that it is a combination of simpler continuous fns

$(f+g)(x) = f(x) + g(x)$, $\mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$
$(fg)(x) = f(x)g(x)$, $\mathrm{dom}(fg) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $\mathrm{dom}\left(\frac{f}{g}\right) = \mathrm{dom}(f) \cap \mathrm{dom}(g) \cap \{x : g(x) \neq 0\}$
$(g \circ f)(x) = g(f(x))$, $\mathrm{dom}(g \circ f) = \mathrm{dom}(f) \cap \{x : f(x) \in \mathrm{dom}(g)\}$

<u>Thm</u> (sum, product, quotient cts fns):
If f and g are continuous at $x_0 \in \mathbb{R}$, then
(a) $f+g$ is continuous at x_0
(b) fg is continuous at x_0
(c) $\frac{f}{g}$ is continuous at x_0, provided $g(x_0) \neq 0$.

<u>Thm</u>: (composition of cts fns) Suppose f is continuous at x_0 and g is continuous at $f(x_0)$. Then $g \circ f$ is continuous at x_0.

<u>Def</u> (bounded function): f is bounded on $S \subseteq \mathrm{dom}(f)$ if there exists $M > 0$ s.t. $|f(x)| \leq M$ for all $x \in S$.
We say f is bounded if f is bounded on $\mathrm{dom}(f)$.

MAJOR THM #1: Archimedean Property
#2: $\mathbb{Q}$ is dense in $\mathbb{R}$
#3: All bounded, monotone seq. converge
#4: Cauchy iff convergent

#5: Bolzano Weierstrass

MAJOR THEOREM 6

<u>Thm</u> (cts fns attain max and min): A continuous function f on a closed interval $[a,b] \subseteq \mathrm{dom}(f)$ attains its maximum and minimum.

$\max(\{f(x) : x \in [a,b]\})$ $\qquad$ $\min(\{f(x) : x \in [a,b]\})$

In particular...

(i) its max and min exist (so f is bounded)

(ii) $\exists$ x_{max} (maximizer), x_{min} (minimizer) $\in [a,b]$ so that

$$\underbrace{f(x_{min})}_{\substack{= \text{ minimum of } f \text{ on } [a,b] \\ = \min(\{f(x) : x \in [a,b]\})}} \le f(x) \le \underbrace{f(x_{max})}_{\substack{= \text{ maximum of } f \text{ on } [a,b] \\ = \max(\{f(x) : x \in [a,b]\})}} \text{ for all } x \in [a,b]$$