Lecture 16

Now onto second key property of continuous functions!

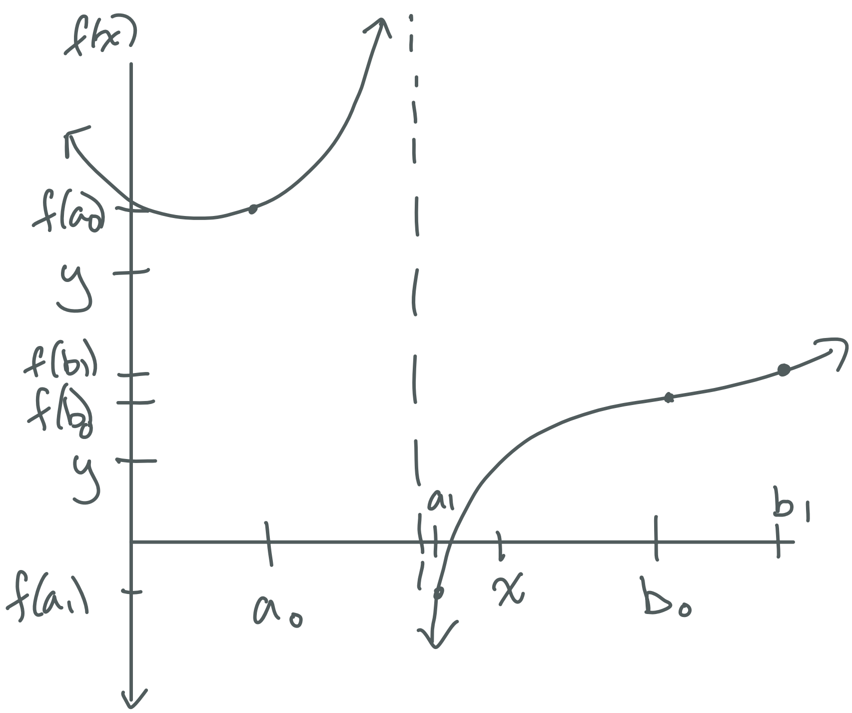
MAJOR THEOREM 7

Thm (Intermediate Value Theorem): If f is continuous on an interval $I \subseteq \text{dom}(f)$, then for all $a, b \in I$, if y lies between $f(a)$ and $f(b)$, then there exists x between a and b s.t. $f(x) = y$.

either $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$

either $a \leq x \leq b$ or $b \leq x \leq a$

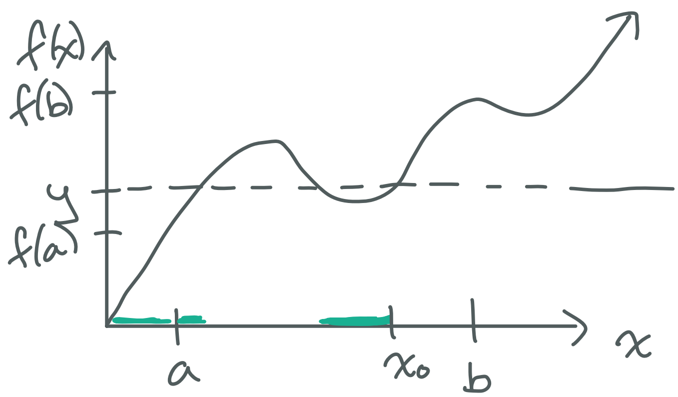
Ex:



Pf: Fix $a, b \in I$. We may assume $a \leq b$. Suppose y lies between $f(a)$ and $f(b)$.

Case 1: $f(a) \leq y \leq f(b)$

We aim to show $\exists\, x \in [a,b]$ s.t. $f(x) = y$.

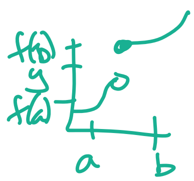


Define $S = \{x \in [a,b] : f(x) \leq y\}$ and let $x_0 = \sup(S)$. For all $n \in \mathbb{N}$, $x_0 - \frac{1}{n}$ is not an upper bound S, so there exists $x_n \in S$ satisfying $x_0 - \frac{1}{n} \leq x_n \leq x_0$. By the Squeeze Lemma, $\lim_{n\to\infty} x_n = x_0$. Furthermore, since $x_n \in S$, $f(x_n) \leq y$. Thus $f(x_0) = \lim_{n\to\infty} f(x_n) \leq y$.

It remains to show $f(x_0) \geq y$. If $x_0 = b$, by assumption $f(x_0) = f(b) \geq y$, and we have proved the result. If $x_0 < b$, define $t_n = \min\{b, x_0 + \frac{1}{n}\}$. By definition $x_0 < t_n \leq x_0 + \frac{1}{n}$, so the Squeeze Lemma ensures $\lim_{n\to\infty} t_n = x_0$. Since $t_n > x_0 = \sup(S)$, $t_n \notin S$, so $f(t_n) > y$ for all n. Since f is continuous, $y \leq \lim_{n\to\infty} f(t_n) = f(x_0)$.

Case 2: $f(a) \geq y \geq f(b)$

Consider the continuous function $-f$. Then $-f(a) \leq -y \leq -f(b)$. By Case 1, there exists $x_0 \in [a,b]$ s.t. $-f(x_0) = -y$. Thus $f(x_0) = y$. $\square$



Morals:

① continuous functions don't have "jumps"

② a continuous function always attains intermediate values