Lecture 1

Math 117
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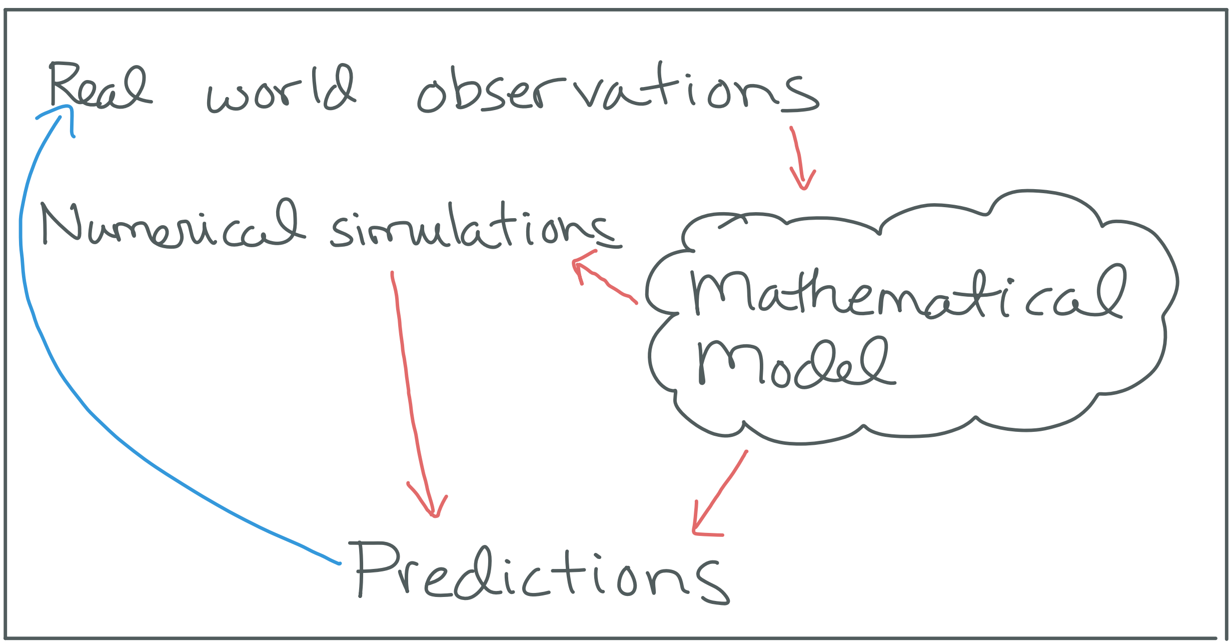
Course goal: transition to higher level math
- "What is a proof?" ⇒ "Let's prove interesting things!"
- This is a mathematical writing course.
  ↳ You must back up your claims using clear, logical arguments.
  ↳ You must be able to precisely state important definitions and theorems.
- If something doesn't make sense...
  1. Carefully read all relevant definitions and theorems. Get the textbook!
  2. Ask me, TA, or LAs for help.
  3. Hang in there. If you stay on top of learning definitions and theorems, things will start to make sense. If you don't, things will become more confusing.

Why analysis? What is analysis?

- take everything you learned in Calculus and put it on rigorous mathematical footing
- analysis is the mathematics of <u>approximation</u>, a key link between mathematics in the real world and mathematics in the fake world



<u>Numbers!</u>

Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

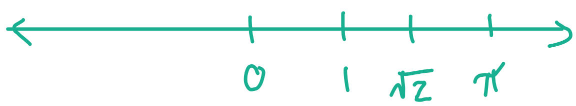
Integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\}$

Real numbers $\mathbb{R} =$

Goal of first part of course: define the real numbers, so we know what to put here

Intuitively: $\mathbb{R}$ is all numbers on number line



Key properties of these classes of numbers

Property 1: Inductive Characterization of $\mathbb{N}$

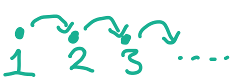
If a subset $S \subseteq \mathbb{N}$ satisfies

(i) $1 \in S$

(ii) if $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$.

Mental picture: "chain reaction"



Ex: $S = \{2, 3, 4, 5\}$ fails (i) and (ii)

IMPORTANT REMARK

This is the basis of proof by induction.

- Suppose $\{P_1, P_2, P_3, \dots\} = \{P_k : k \in \mathbb{N}\}$ is a list of statements.

  P_k = $k+2$ is an integer

  P_k = Katy wants to eat k cookies

- Suppose you can prove that

  (a) P_1 is true ← base case

  (b) For all $n \in \mathbb{N}$, $\underbrace{\text{if } P_n \text{ is true}}_{\text{inductive hypothesis}}$, then P_{n+1} is true.

inductive step

What does this tell us about $S = \{k \in \mathbb{N} : P_k \text{ is true}\}$?

Condition (a) ensures $1 \in S$.

Condition (b) ensures that, if $n \in S$, then $n+1 \in S$.

Therefore, Property 1 ensures that $S = \mathbb{N}$.

In other words, if you can prove criteria (a) and (b), then P_k is true for all $k \in \mathbb{N}$.

The previous remark shows that the fact that proof by induction works is a consequence of Property 1 of $\mathbb{N}$.

Exercise 1

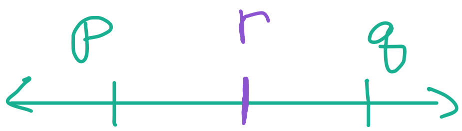
Prove by induction that, for all $n \in \mathbb{N}$,

$$1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.$$

Property 2: $\mathbb{Q}$ is dense in $\mathbb{Q}$

Prop: For any $p, q \in \mathbb{Q}$ with $p < q$, there exists $r \in \mathbb{Q}$ satisfying $p < r < q$.

"Between any two rational numbers, there is a rational number"



Pf: Let $r = \frac{p+q}{2}$.

First, we show $r \in \mathbb{Q}$. Since $p, q \in \mathbb{Q}$, $\exists\, m_p, n_p, m_q, n_q \in \mathbb{Z}$ with $n_p \neq 0$ and $n_q \neq 0$ so that $p = \frac{m_p}{n_p}$ and $q = \frac{m_q}{n_q}$.

Thus, $r = \frac{p+q}{2} = \frac{1}{2}\left(\frac{m_p}{n_p} + \frac{m_q}{n_q}\right) = \frac{\overbrace{m_p n_q + m_q n_p}^{\text{integer}}}{\underbrace{2 n_p n_q}_{\text{nonzero integer}}}$,

so $r \in \mathbb{Q}$.

Furthermore, since $p<q$, so $\frac{p}{2}<\frac{q}{2}$.
So,

$$p=\frac{p}{2}+\frac{p}{2}<\underbrace{\frac{p}{2}+\frac{q}{2}}_{r}<\frac{q}{2}+\frac{q}{2}=q.$$

Property 3: Absolute Value and Distance

We may take the absolute value of any number in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Def: $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Thm (basic properties of $|\cdot|$): For all $a, b \in \mathbb{R}$,

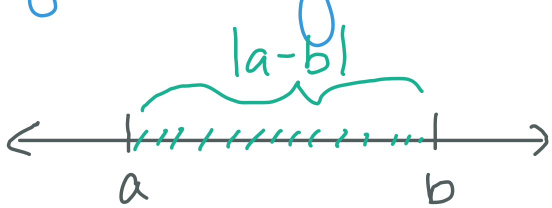
(i) $|a| \geq 0$

(ii) $|ab| = |a||b|$ ← "absolute value distributes over multiplication"

(iii) $|a+b| \leq |a| + |b|$ ← Triangle inequality

We can use the absolute value to define a notion of distance between any two elements of $\mathbb{R}$.

Def: For any $a, b \in \mathbb{R}$, $\operatorname{dist}(a,b) = |a-b|$.



Property 4 : Strict Containment

follow directly from definition

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \underbrace{\mathbb{Q} \subsetneq \mathbb{R}}_{\text{we will show next}}$

strictly contained

Recall: Given two sets A, B

- $A \subseteq B$ if $a \in A$ implies $a \in B$
- $A \nsubseteq B$ if there exists $a \in A$ for which $a \notin B$.
- $A \subsetneq B$ if $A \subseteq B$ but $A \neq B$.