Lecture 2: Highlights

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Office Hours

Numbers!

Natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$

Integers $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

Rational numbers $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$

Real numbers $\mathbb{R} =$ what to put here?

Property 1: Inductive Characterization of $\mathbb{N}$

If a subset $S \subseteq \mathbb{N}$ satisfies

(i) $1 \in S$

(ii) if $n \in S$, then $n+1 \in S$

then $S = \mathbb{N}$.

Property 2: $\mathbb{Q}$ is dense in $\mathbb{Q}$

Prop: For any $p, q \in \mathbb{Q}$ with $p < q$, there exists $r \in \mathbb{Q}$ satisfying $p < r < q$.

Property 3: Absolute Value and Distance

Def: $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$

Thm (basic properties of $|\cdot|$): For all $a, b \in \mathbb{R}$,

(i) $|a| \geq 0$

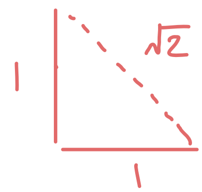
(ii) $|ab| = |a||b|$ ← the absolute value can be distributed over multiplication

(iii) $|a+b| \leq |a| + |b|$ ← Triangle inequality

Property 4: Strict Containment

$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}$

Prop: $\sqrt{2} \notin \mathbb{Q}$



Moral: there are useful numbers missing from $\mathbb{Q}$.

Def: (field): A set F is a field if it has two operations (addition and multiplication) that satisfy the following properties $\forall a, b, c \in F$:

(A1) $a+(b+c) = (a+b)+c$ associativity

(A2) $a+b = b+a$ commatativity

(A3) $\exists$ an element in F called 0 s.t. $\forall a \in F$, $a+0=a$ identity

(A4) for each $a \in F$, $\exists$ an element called $-a \in F$ s.t. $a+(-a)=0$ inverse

(m1) $a(bc) = (ab)c$ associativity

(m2) $ab=ba$ commatativity

(m3) $\exists$ an element in F called 1 s.t. $1 \neq 0$ and $\forall a \in F$, $a \cdot 1 = a$ identity

(m4) for each $a \in F$, $a \neq 0$, $\exists$ an element called $\frac{1}{a}$ s.t. $a \cdot \frac{1}{a} = 1$. inverse

(DL) $a(b+c) = ab+ac$ distributive law

Remark: $\mathbb{N}$ and $\mathbb{Z}$ aren't fields
$\mathbb{Q}$ is a field
$M_n(\mathbb{R})$ isn't a field for $n \geq 2$

Using the definition of a field, you can rigorously prove familiar algebraic properties.

<u>Thm</u>: If F is a field, then $\forall\, a,b \in F$:

(i) If $a+c=b+c$, then $a=b$

(ii) $a \cdot 0 = 0$