Lecture 3

Last time, we defined what it meant for a set F to be a field.

We will focus on a specific type of field known as an ordered field.

Def (ordered field): A field F is an ordered field if it has an ordering relation $\leq$ so that, for all $a, b, c \in F$:

(O1) either $a \leq b$ or $b \leq a$ — totality

(O2) if $a \leq b$ and $b \leq a$, then $a = b$ — antisymmetry

(O3) if $a \leq b$ and $b \leq c$, then $a \leq c$ — transitivity

(O4) if $a \leq b$, then $a + c \leq b + c$ — addition

(O5) if $a \leq b$ and $c \geq 0$, then $ac \leq bc$ — multiplication

Def: Given an ordered field F and $a, b \in F$, if $a \leq b$ and $a \neq b$, then write $a < b$.

On an ordered field, we can define the notion of maximum or minimum of a set.

Def (maximum, minimum): Suppose $S \subseteq F$, where F is an ordered field.

- If there exists $s_0 \in S$ satisfying $s_0 \geq s$ for all $s \in S$, then s_0 is the <u>maximum</u> of S and write $s_0 = \max(S)$.
  "s_0 is the largest element in the set"
- If there exists $s_0 \in S$ satisfying $s_0 \leq s$ for all $s \in S$, then s_0 is the <u>minimum</u> of S and write $s_0 = \min(S)$.
  "s_0 is the smallest element in the set"

Ex: Let F be an ordered field.
Any finite set $S = \{s_1, s_2, \ldots, s_n\} \subseteq F$ has a maximum and minimum.

Let $F = \mathbb{Q}$. Fix $a, b \in \mathbb{Q}$ with $a < b$.
If $S = \mathbb{N}$, $\min(S) = 1$, $\max(S)$ D.N.E. does not exist
If $S = \{q \in \mathbb{Q} : a \leq q < b\}$, $\min(S) = a$

<u>Claim</u>: $\max(S)$ D.N.E.

Pf of Claim: Assume, for the sake of contradiction, that $s_0 = \max(S)$. Since $s_0 \in S$, so $a \leq s_0 < b$. Since $\mathbb{Q}$ is dense in $\mathbb{Q}$, $\exists\, r \in \mathbb{Q}$ so that $s_0 < r < b$. Thus $r \in S$. This contradicts that $s_0 = \max(S)$.

Likewise, on an ordered field, we can define what it means for a set to be bounded above or below.

Def: (bounded above/below): Suppose $S \subseteq F$ for an ordered field F.

- If there exists $M \in F$ satisfying $s \leq M$ $\forall s \in S$, then S is <u>bounded above</u> and M is an <u>upper bound</u> of S.
- If there exists $m \in F$ satisfying $s \geq m$ $\forall s \in S$, then S is <u>bounded below</u> and m is a <u>lower bound</u> of S.
- If S is bounded above and bounded below, then S is <u>bounded</u>.

Mental picture: 

Ex: $F = \mathbb{Q}$, $a, b \in \mathbb{Q}$, $a < b$

$S = \{q \in \mathbb{Q} : a \leq q < b\}$ is bounded

$S = \mathbb{Z}$ is not bounded

NOTE: Unlike the maximum of a set, the upper bound of a set doesn't need to belong to the set.

What about when a set "almost" has a maximum?

<u>Def</u> (supremum / infimum): Consider an ordered field F.

- If $S \subseteq F$ is bounded above and there exists $M_0 \in F$ satisfying...
  - (a) M_0 is an upper bound of S
  - (b) if M is an upper bound of S, then $M_0 \le M$

  we say M_0 is the <u>supremum</u> of S and write $M_0 = \sup(S)$.

  "M_0 is the least upper bound"

- If $S \subseteq F$ is bounded below and there exists $m_0 \in F$ satisfying...
  - (a) m_0 is a lower bound of S
  - (b) if m is a lower bound of S, then $m_0 \ge m$

  we say m_0 is the <u>infimum</u> of S and write $m_0 = \inf(S)$.

  "m_0 is the greatest lower bound"

Ex: $F = \mathbb{Q}$, $a, b \in \mathbb{Q}$, $a < b$

$S = \{q \in \mathbb{Q} : a \leq q < b\}$

<u>Claim</u>: $\sup(S) = b$

<u>Pf</u>: By definition of S, b is an upper bound of S. Fix $M \in \mathbb{Q}$ s.t. $s \leq M$ $\forall s \in S$. It suffices to show $b \leq M$.

Since $s \leq M$ $\forall s \in S$, $a \leq M$.

Assume, for the sake of contradiction, that $b > M$. By density of $\mathbb{Q}$ in $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ s.t. $M < r < b$. Thus $r \in S$. This contradicts that M is an upper bound of S. Therefore $b \leq M$.

We conclude $b = \sup(S)$. $\square$

prove on HW

$\downarrow$

<u>Thm</u>: Consider an ordered field F and $S \subseteq F$.

(i) If $\max(S)$ exists, then $\sup(S) = \max(S)$.

(ii) If $\min(S)$ exists, then $\inf(S) = \min(S)$.

<u>Moral:</u> The notion of supremum is a generalization of the notion of maximum.

<u>Def:</u> (real numbers): The set of real numbers is the ordered field containing $\mathbb{Q}$ with the property that every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above has a supremum.

"The Least Upper Bound Property of $\mathbb{R}$"

<u>Thm:</u> The real numbers exists.