Lecture 3 - Math movie competition

Def (ordered field): A field F is an ordered field if it has an ordering relation $\leq$ so that, for all $a,b,c \in F$:

(O1) either $a \leq b$ or $b \leq a$ — totality
(O2) if $a \leq b$ and $b \leq a$, then $a = b$ — antisymmetry
(O3) if $a \leq b$ and $b \leq c$, then $a \leq c$ — transitivity
(O4) if $a \leq b$, then $a + c \leq b + c$ — addition
(O5) if $a \leq b$ and $c \geq 0$, then $ac \leq bc$ — multiplication

Def: Given an ordered field F and $a,b \in F$, if $a \leq b$ and $a \neq b$, then write $a < b$.

Def (maximum, minimum): Suppose $S \subseteq F$, where F is an ordered field.

- If there exists $s_0 \in S$ satisfying $s_0 \geq s$ for all $s \in S$, then s_0 is the <u>maximum</u> of S and write $s_0 = \max(S)$.
  "s_0 is the largest element in the set"
- If there exists $s_0 \in S$ satisfying $s_0 \leq s$ for all $s \in S$, then s_0 is the <u>minimum</u> of S and write $s_0 = \min(S)$.
  "s_0 is the smallest element in the set"

Def: (bounded above/below): Suppose $S \subseteq F$ for an ordered field F.

- If there exists $M \in F$ satisfying $s \leq M$ $\forall s \in S$, then S is <u>bounded above</u> and M is an <u>upper bound</u> of S.
- If there exists $m \in F$ satisfying $s \geq m$ $\forall s \in S$, then S is <u>bounded below</u> and m is a <u>lower bound</u> of S.
- If S is bounded above and bounded below, then S is <u>bounded</u>.

Def (supremum/infimum): Consider an ordered field F.

- If $S \subseteq F$ is bounded above and there exists $M_0 \in F$ satisfying...
  - (a) M_0 is an upper bound of S
  - (b) if M is an upper bound of S, then $M_0 \le M$

  we say M_0 is the <u>supremum</u> of S and write $M_0 = \sup(S)$.

  "M_0 is the least upper bound"

- If $S \subseteq F$ is bounded below and there exists $m_0 \in F$ satisfying...
  - (a) m_0 is a lower bound of S
  - (b) if m is a lower bound of S, then $m_0 \ge m$

  we say m_0 is the <u>infimum</u> of S and write $m_0 = \inf(S)$.

  "m_0 is the greatest lower bound"

Thm: Consider an ordered field F and $S \subseteq F$.

(i) If $\max(S)$ exists, then $\sup(S) = \max(S)$.

(ii) If $\min(S)$ exists, then $\inf(S) = \min(S)$.

<u>Moral</u>: The notion of supremum is a generalization of the notion of maximum.

<u>Def</u>: (real numbers): The set of real numbers is the ordered field containing $\mathbb{Q}$ with the property that every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above has a supremum.

"The Least Upper Bound Property of $\mathbb{R}$"

<u>Thm</u>: The real numbers exists.

Ex: $F = \mathbb{Q}$, $a, b \in \mathbb{Q}$, $a < b$

$S = \{q \in \mathbb{Q} : a \leq q < b\}$

<u>Claim</u>: $\sup(S) = b$

<u>Pf</u>: By definition of S, b is an upper bound of S. Fix $M \in \mathbb{Q}$ s.t. $s \leq M$ $\forall s \in S$. It suffices to show $b \leq M$.

Since $s \leq M$ $\forall s \in S$, $a \leq M$.

Assume, for the sake of contradiction, that $b > M$. By density of $\mathbb{Q}$ in $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ s.t. $M < r < b$. Thus $r \in S$. This contradicts that M is an upper bound of S. Therefore $b \leq M$.

We conclude $b = \sup(S)$. $\square$

prove on HW

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