Lecture 4

We'll study two major theorems for $\mathbb{R}$:
Archimedean Property and $\mathbb{Q}$ is dense in $\mathbb{R}$.

MAJOR RESULT #1

Thm (Archimedean Property): If $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b > 0$, then there exists $n \in \mathbb{N}$ so that $na > b$.

spoon bathtub

Remark: Even if a is really small and b is huge, some integer multiple of a is bigger than b.

"Given enough time, one can empty a large bathtub with a small spoon."

We will prove by contradiction.

Scratchwork:

$$P = [\forall a, b > 0, \exists n \in \mathbb{N} \text{ s.t. } na > b]$$

$$\neg P = [\exists a, b > 0 \text{ s.t. } \forall n \in \mathbb{N}, na \leq b]$$

Pf: Assume, for the sake of contradiction, that $\exists\ a, b \in \mathbb{R}$ with $a > 0$, $b > 0$ s.t. for all $n \in \mathbb{N}$, $na \leq b$.

$\{a, 2a, 3a, 4a, \ldots\}$

Define $S = \{na : n \in \mathbb{N}\}$, so b is an upper bound for S. Since S is a nonempty subset of $\mathbb{R}$ that is bounded above, by defn of $\mathbb{R}$, S has a supremum. Define $s_0 = \sup(S)$.

Since $a > 0$, we have $s_0 - a < s_0 < s_0 + a$. Since $s_0 = \sup(S)$, there exists $n_0 \in \mathbb{N}$ s.t. $s_0 - a < n_0 a \Rightarrow s_0 < (n_0 + 1)a$.

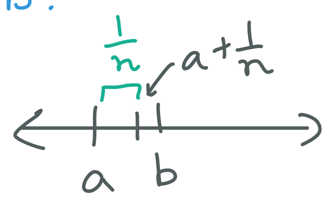
Since $(n_0+1)a \in S$, this contradicts the fact that s_0 is an upper bound of S. $\square$

As a consequence of the Archimedean Property, we have a few useful lemmas...

Lemma: For any $a \in \mathbb{R}$, there exists $n \in \mathbb{N}$ s.t. $a < n$.

Pf: If $a \leq 0$ then the result holds for $n = 1$.
If $a > 0$ ← bathtub, then since $1 > 0$ ← spoon, by A.P. there exists $n \in \mathbb{N}$ s.t. $1 \cdot n > a$. $\square$

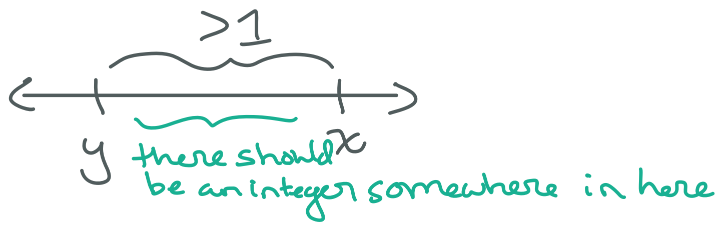
Lemma: For any $a, b \in \mathbb{R}$, $a < b$, there exists $n \in \mathbb{N}$ so that $a + \frac{1}{n} < b$.

Mental image: 

Pf: Let $y = b - a > 0$ (spoon) and $1 > 0$ (bathtub). By A.P., there exists $n \in \mathbb{N}$ s.t. $ny > 1 \Leftrightarrow y > \frac{1}{n}$
$\Leftrightarrow b - a > \frac{1}{n} \Leftrightarrow a + \frac{1}{n} < b$. $\square$

Lemma: If $x, y \in \mathbb{R}$ satisfy $1 < x - y$, then $\exists\, m \in \mathbb{Z}$ so that $y < m < x$.

Mental image:



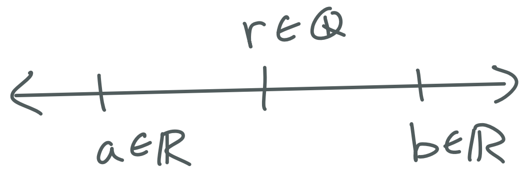
Pf: By the first lemma, there exists $n \in \mathbb{N}$ s.t. $n > y$. Define $S = \{ j \in \mathbb{Z} : y < j \le n \}$. Then S is nonempty and finite, so $m = \min(S)$ exists. By defn of m, $m \in \mathbb{Z}$, $y < m$, and $m - 1 \le y$. Therefore, $y < m \le 1 + y < x$. $\square$

Now, we will apply the previous lemmas to prove...

MAJOR THEOREM #2

Thm ($\mathbb{Q}$ is dense in $\mathbb{R}$): If $a, b \in \mathbb{R}$ with $a<b$, there exists $r \in \mathbb{Q}$ satisfying $a<r<b$.

Mental image:

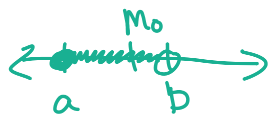


This is similar to the result we proved on the first day that between any two rational numbers there is a rational number.

Pf: By the lemma, $\exists\ n \in \mathbb{N}$ s.t. $a+\frac{1}{n}<b \iff na+1<bn \iff 1<bn-an$. By the other lemma, there exists $m \in \mathbb{Z}$ so that $an<m<bn \iff a<\frac{m}{n}<b$. $\square$

We now have all the tools we need to rigorously prove our previous claims about the minimum/maximum/infimum/supremum of subsets of $\mathbb{R}$! For example...

Prop: For $a, b \in \mathbb{R}$, $a < b$, the set $S = [a, b)$ does not have a maximum and $\sup(S) = b$.



Pf:
First, we show that S does not have a maximum. Assume, for the sake of contradiction, that $\max(S) = M_0$. Since $M_0 \in S$, $a \le M_0 < b$. By density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, r \in \mathbb{Q}$ s.t. $M_0 < r < b$, so $r \in S$. This contradicts that M_0 was the largest element in S.

Now, we show $\sup(S) = b$. By def'n of S, b is an upper bound. Suppose M_0 is another upper bound of S. (so $M_0 \ge a$.) If $M_0 < b$, then by density of $\mathbb{Q}$ in $\mathbb{R}$, $\exists\, r \in \mathbb{Q}$ s.t. $M_0 < r < b$, so $r \in S$, which is a contradiction. Thus, $M_0 \ge b$, so b is the least upper bound. $\square$

Going forward, we will use $+\infty$ and $-\infty$ to simplify our notation for suprema and infima.

Ex: $(a,+\infty) = \{x \in \mathbb{R} : a < x\} = \{x \in \mathbb{R} : a < x < +\infty\}$

Def (Unbounded above/below): Suppose $S \subseteq \mathbb{R}$ is nonempty.
If S is not bounded above, write $\sup(S) = +\infty$.
If S is not bounded below, write $\inf(S) = -\infty$.

Remark: Given a nonempty $S \subseteq \mathbb{R}$,

- By defn of supremum and $\mathbb{R}$

$$\begin{aligned} S \text{ has a supremum} &\iff S \text{ is bounded above} \\ &\iff \sup(S) \in \mathbb{R} \end{aligned}$$

- Similarly,

$$\begin{aligned} S \text{ doesn't have a supremum} &\iff S \text{ is not bounded above} \\ &\iff \sup(S) = +\infty \end{aligned}$$

Using this notation, even though not every set has a supremum, for any nonempty $S \subseteq \mathbb{R}$, $\sup(S)$ has meaning.